Chromatic and Space Charge Effects in Nonlinear Integrable Optics

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Outline

• Crash Survey of Integrable Optics

• Dispersion & Chromaticity

• Space Charge & Invariants

• Future work
Crash Survey of Integrable Optics
The properties of linear strong focusing

• Strong focusing is robust because it is integrable

– Two transverse Courant-Snyder invariants

\[ I = \beta p^2 + 2\alpha qp + \gamma q^2 \]

  • orbits are integrable — regular, bounded, periodic motion
  • KAM theorem notably does not apply to linear systems

– KAM Th\textsuperscript{m} does not apply to linear systems

  • single tune makes whole system unstable to resonant perturbations
  • higher-order effects such as chromaticity restore some stability

– Linearity leaves system susceptible to parametric resonances

  • core-halo
  • resistive wall instability
  • beam break-up
  • ...

...
Additional stability from nonlinear integrable optics

• Key ideas:

  – A system with large tune spread…
    • fast Landau damping
    • suppresses parametric resonances
    • promises beam transport with lower losses

  – … but integrable dynamics
    • KAM Thm provides stability
    • on-momentum orbits are bounded and regular
    • perturbations lead to resonant lines…
    • …but orbits must diffuse out of dynamic aperture

  – so we expect stable beam dynamics in space charge
Conditions for Integrability

- Bertrand-Darboux equation

\[ xy \left( \partial_x^2 - \partial_y^2 \right) U + (y^2 - x^2 + c^2) \partial_{xy} U + 3(y\partial_x - x\partial_y) U = 0 \]

- Hamiltonians with 2\textsuperscript{nd} invariants quadratic in momentum satisfy:
  - differential equation is \textit{linear}
  - any superposition of potentials that satisfy this differential equation will have a 2\textsuperscript{nd} invariant and be integrable

- Other auxiliary conditions for accelerators:
  - matched beta functions in the drifts with these nonlinear elements
  - equal vertical and horizontal linear tunes
Nonlinearities suppress parametric resonances
Dispersion & Chromaticity
Dispersion & Chromaticity I

- Off-momentum particles couple motion to energy

  - Linear lattice chromaticity:
    - energy-dependent tune could cross nonlinear resonance
    - no loss of integrability (assuming linear RF bucket/coasting beam)

  - Linear lattice dispersion:
    - large dispersion can cause large beam size

- Potential problems for elliptic potential
  - unequal tunes violates the Bertrand-Darboux equation
  - dispersion violates the equal beta function requirement

- Conclusions:
  - defocussing quadratic perturbation due to differing chromaticities
  - already have large tune spreads — no need to remove all the chromaticity
Single-turn Map

\[ \mathcal{M} = e^{-:H:C} \]

Figure from S. Nagaitsev, “IOTA Physics Goals” (2012)
Single-turn Map

\[ M_{1 \rightarrow 2}(z_1) = z_2 \]
Dispersion & Chromaticity II

- Computed for the continuously varying magnet

- Details in extra slides…
- Compute single-turn map as

\[
M_{\text{IOTA}} = \mathcal{A}^{-1} \exp \left\{ -: \frac{t}{1 - \delta} \int_{\ell/2}^{\ell} ds \ U(\bar{x} - \eta(s)\delta, \bar{y}):: \right\} e^{-:\overline{H}}: \exp \left\{ -: \frac{t}{1 - \delta} \int_{0}^{\ell/2} ds \ U(\bar{x} - \eta(s)\delta, \bar{y}):: \right\} \mathcal{A}
\]

- and the related Hamiltonian

\[
\overline{H} = \frac{\mu_0}{2} \left\{ [1 - C_x(\delta)] (p_x^2 + \bar{x}^2) + [1 - C_y(\delta)] (p_y^2 + \bar{y}^2) + \frac{t}{1 - \delta} \int_{0}^{\ell_{\text{drift}}} U(\bar{x} - \eta(s') \delta, \bar{y}) ds' \right\} + \text{h.o.t.}
\]

- For the Bertrand-Darboux potential, we require:
  - Very particular form for U
  - equal vertical and horizontal linear tunes
Dispersion & Chromaticity III

• New Set of Design Rules:
  
  – Twiss parameters
    • require equal beta functions to get desired cancellation
    • effective double-focusing lens for on-momentum linear map
  
  – Chromaticity
    • transverse tunes must be equal
    • familiar chromaticity correction schemes sufficient
    • correct to make $C_x = C_y$
  
  – Dispersion
    • dispersion modifies the integrable potential
    • drift section for elliptic magnets must be dispersion-free
Space Charge & Invariants
Presence of Space Charge Changes Distribution
Presence of Space Charge Changes Distribution
Presence of Space Charge Changes Distribution
But the transverse beam distribution is static...

After 700+ turns, transverse phase space remains static

Transverse beam size has initial growth, followed by very small variations
What’s going on?

- Hamiltonian now contains self-consistent space charge
  - Hamiltonian given by

\[
H = H_0(J) + \int dJ' d\psi' G(J, \psi; J', \psi') f(J')
\]

- \([G] \propto [\text{current}]\)
  - intensity-dependent effects induce diffusion
  - distribution diffuses to fill “potential”
  - achieves steady state through space charge induced stochasticity

- Speculation, requires better evidence
  - diffusion rate \(\propto\) current
  - actual calculation (unlikely)

\(^1\)see, e.g., Lichtenberg & Lieberman, §5.4
What’s going on?

- **Diffusion in Action Space**
  - Fokker-Planck Equation for perturbed integrable systems\(^1\)

  \[
  \frac{\partial f}{\partial t} = \frac{\partial}{\partial J} \left[ \frac{1}{4} (\Delta t)^2 \left( \frac{\partial}{\partial J} \left\langle \left( \frac{\partial H}{\partial \psi} \right)^2 \right\rangle_\psi \right) \frac{\partial f}{\partial J} \right]
  \]

  - Modified Hamiltonian with space charge

  \[
  H = H_0(J) + \int dJ' g_0(J, J') f(J') + \sum_{n \neq 0} g_n(J, J') e^{in\psi} f(J')
  \]

  - **Particles drift in effective potential and diffuse**
    - some steady state reached
    - some particles diffuse out of the potential well
    - complicated by resonance islands, correlations, nonlinear Fokker-Planck…

\(^1\)see, e.g., Lichtenberg & Lieberman, §5.4
Future Work

• What do the chromaticity corrections do to the invariants?
  – What do sextupoles, etc., do to the dynamic aperture?
  – How to minimize the impact on the beam?
  – What is the diffusion time for particles on resonance?

• What does space charge do?
  – How does space charge affect the invariants?
  – What can be done to compensate space charge?
  – Is there a collective invariant that remains?
Thank you for your attention

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Digression on Lie Operators

• Lie operators from Poisson brackets

\[ \dot{z} = -\{H, z\} \rightarrow \dot{z} = -:H:z \]

\[ z(t) = e^{-:H:t}z(0) \]

– Advantages
  • can multiply maps, cannot multiply Hamiltonians
  • maps make coordinate transformations into similarity transformations

– Disadvantages
  • a lot of formalism to get to the physics
  • difficult to work with time-varying Hamiltonians

• Key Identities

  • BCH Identity
    \[ e^{C} = e^{A}e^{B} \quad C = A + B + \frac{1}{2} :A:B + \frac{1}{12} (:A^{2}B + :B^{2}A) + \ldots \]

  • Similarity transformation
    \[ e^{A}e^{B}e^{-A} = \exp (:e^{A}:B:) \]
When are sextupoles optically transparent?

• Lie operator approach

\[ M = e^{-S_n : z^n :} e^{-: h_2 :} e^{-S_n : z^n :} \]

\[ e^{-: h_2 :} = A^{-1} \underbrace{R(\theta)}_{\text{pure rotation}} A \]

\[ M = e^{-: h_2 :} \exp(-S_n : e^{: h_2 :} z^n :) \exp(-S_n : z^n :) \]

\[ M = A^{-1} R \exp(-S_n : R(\bar{z})^n :) \exp(-S_n : \bar{z}^n :) \]

\[ R \propto -1, \theta = (2n + 1)\pi \implies \exp(-S_n : R(\bar{z})^n :) \exp(-S_n : \bar{z}^n :) A = 1 \]

• Off-momentum particles do not cancel exactly because \( \theta \) is energy-dependent. This is the basis of chromaticity correction.
When are sextupoles optically transparent?

- Pictorial approach (design momentum)
When are sextupoles optically transparent?

- Pictorial approach (off-momentum)
The horror...

\[
\mathcal{M} = \left( \prod_{i=0}^{N/2} \exp \left\{ -\frac{p^2}{2} + t_i \mathcal{V}_i(x, y): \Delta s \right\} \right) e^{-\hbar_0} \left( \prod_{i=N/2}^{N} \exp \left\{ -\frac{p^2}{2} + t_i \mathcal{V}_i(x, y): \Delta s \right\} \right)
\]

\[
= \left( \prod_{i=0}^{N/2} \exp \left\{ -t_i : e^{-\frac{p^2/2}{(i+1/2)}: \Delta s} \mathcal{V}_i(x, y): \Delta s \right\} \right) \circ \underbrace{\overbrace{\left( e^{-\frac{p^2/2}{\epsilon/2}} e^{-\hbar_0} e^{-\frac{p^2/2}{\epsilon/2}} \right)}_{e^{-\hbar_2}} \circ \left( \prod_{i=N/2}^{N} \exp \left\{ -t_i : e^{\frac{p^2/2}{(i+1/2)}: \Delta s} \mathcal{V}_i(x, y): \Delta s \right\} \right)}
\]
\[ e^{-\bar{h}_2} = \mathcal{A} e^{-\bar{h}_2} \mathcal{A}^{-1} \]

\[
\mathcal{A}^{-1} = \begin{pmatrix}
\frac{1}{\sqrt{\beta_x}} & 0 & 0 & 0 & 0 & -\frac{\eta}{\sqrt{\beta_x}} \\
\frac{\alpha_x}{\sqrt{\beta_x}} & \frac{1}{\sqrt{\beta_y}} & 0 & 0 & 0 & 0 \\
0 & \frac{\alpha_x}{\sqrt{\beta_y}} & \frac{1}{\sqrt{\beta_x}} & 0 & 0 & 0 \\
0 & 0 & \frac{\alpha_x}{\sqrt{\beta_y}} & \frac{1}{\sqrt{\beta_y}} & 0 & 0 \\
\eta' & \eta & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

\[ \mathcal{M} = \left( \prod_{i=0}^{N/2} \exp \left\{ -t_i e^{-(i+1/2) \frac{p^2}{2} \Delta s} \mathcal{V}_i(x, y) \Delta s \right\} \right) \circ \left( \mathcal{A} e^{-\bar{h}_2} \mathcal{A}^{-1} \right) \circ \left( \prod_{i=N/2}^{N} \exp \left\{ -t_i e^{(i+1/2) \frac{p^2}{2} \Delta s} \mathcal{V}_i(x, y) \Delta s \right\} \right) \]
The Danilov-Nagaitsev potential normalizing trick as follows:

\[
A^{-1} \exp \left\{ -t_i : e^{(i+1/2) : p^2 : 2V_i(x, y) : \Delta s} \right\} = \\
A^{-1} \exp \left\{ -t_i : e^{(i+1/2) : p^2 : 2V_i(x, y) : \Delta s} \right\} A A^{-1} = \\
A_i^{-1} e^{-(i+1/2) : p^2 / 2 : \Delta s} \exp \left\{ -t_i : V_i(x, y) : \Delta s \right\} e^{(i+1/2) : p^2 / 2 : \Delta s} A
\]

\[
A_0^{(i)} = \begin{pmatrix} \frac{1}{\sqrt{\beta_x}} & 0 & 0 & 0 \\ \alpha_x / \sqrt{\beta_x} & \sqrt{\beta_x} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{\beta_y}} & 0 \\ 0 & 0 & \alpha_x / \sqrt{\beta_y} & \sqrt{\beta_y} \end{pmatrix}
\]

\[
V_i(x, y) = V_i \left( A_0^{(i)}(x, y) \right)
\]
\[ A^{-1} e^{-(i+1/2):p^2/2: \Delta s} \exp \left\{ -t_i : \mathcal{V}_i(x, y): \Delta s \right\} e^{(i+1/2):p^2/2: \Delta s} A = \exp \left\{ -t : \mathcal{V} \left( \bar{x} - \delta \frac{\eta}{\sqrt{\beta_x}}, \bar{y} \right): \Delta s \right\} \]

**Final transfer map in normalized coordinates**

\[
\left( \prod_{i=0}^{N/2} \exp \left\{ - \frac{p^2}{2} + t_i \mathcal{V}_i(x, y): \Delta s \right\} \right) e^{-\hbar_0: \left( \prod_{i=N/2}^{N} \exp \left\{ - \frac{p^2}{2} + t_i \mathcal{V}_i(x, y): \Delta s \right\} \right)} = A \exp \left\{ \sum_i -(1-\delta)t : \mathcal{V} \left( \bar{x} - \delta \frac{\eta_i}{\sqrt{\beta_i}}, \bar{y} \right): \right\} e^{-\hbar_2: \exp \left\{ \sum_i -(1-\delta)t : \mathcal{V} \left( \bar{x} - \delta \frac{\eta_i}{\sqrt{\beta_i}}, \bar{y} \right): \right\}} A^{-1}
\]

\[
\hbar_2 = \frac{\mu_0}{2} \left[ (1 - C_x \delta) (\bar{p}_x^2 + \bar{x}^2) + (1 - C_y \delta) (\bar{p}_y^2 + \bar{y}^2) \right]
\]