Scaling Properties of Resonances in Non-Scaling FFAGs

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Fermilab

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Introduction

- During ramping of an FFAG, betatron tunes cross many nonlinear resonances.
- We study here emittance growth and beam loss crossing the 3rd-order resonance $3\nu_x = \ell$.
- Chao *et al* and Aiba *et al* attempted to derive *trap efficiency* during resonance crossing. There are successes, but do not fit experimental results well.
- We set 20% as *tolerable emittance increase* or 2.5% as *tolerable trap-fraction* in resonance crossing, and derive scaling laws for critical allowable resonance strength.
- The scaling law can be obtained by solving Hamilton's equations of motion by perturbation.
- Will comment on crossing $\nu_x - 2\nu_z = \ell$ resonance.

Model Ring for Simulation

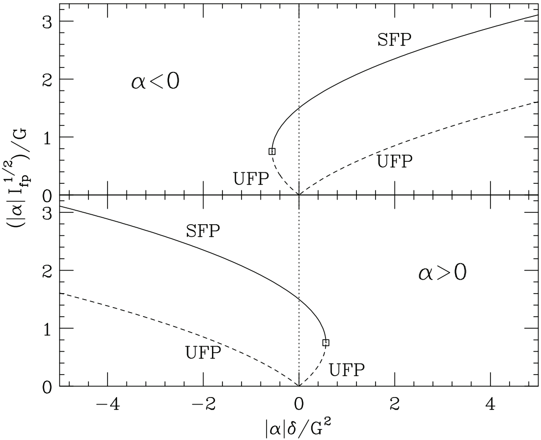
- The model ring for simulation is similar to the Fermilab Booster: $C = 474$ m with 24-fold symmetric FODO lattice.
- Betatron functions are:
  $\beta_{xF} = 40.0$ m, $\beta_{zF} = 8.3$ m
  $\beta_{xD} = 6.3$ m, $\beta_{zD} = 21.4$ m
- A sextupole and octupole at one of the D-quads provide 3rd-order nonlinearity and detuning.
- Kinetic beam energy is fixed at 1 GeV.
- The simulations ramp the horizontal tune:
  ν_x: from 6.4 to 6.24 crossing $3\nu_x = 19$ resonance
  ν_z: fixed at 6.45, but chosen not to excite difference or sum resonance as well as 4th-order resonances.

Fixed Points

- Hamiltonian in the horizontal phase space:

  $H = \delta I + \frac{1}{2}\alpha I^2 + G I^{3/2}\cos 3\psi$, proximity $\delta = \nu - \frac{\ell}{3}$, detuning α.

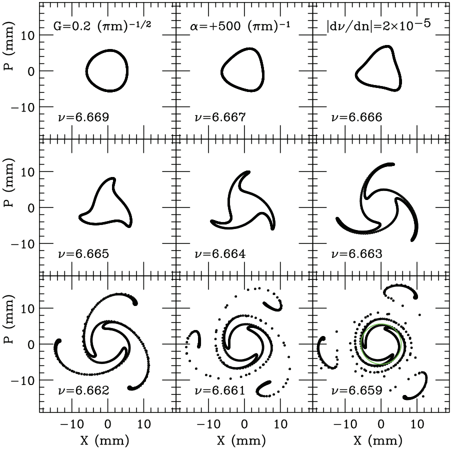
- We study *downward ramping* of horizontal tune ν.

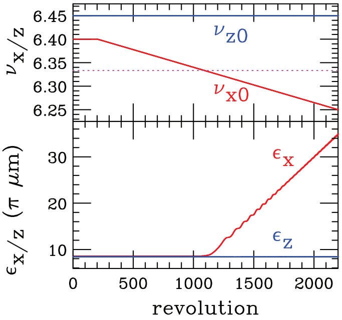


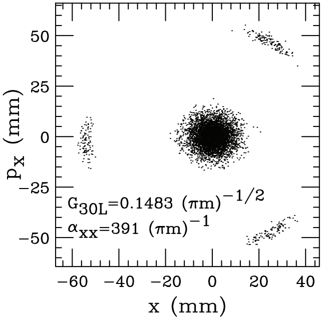
$\longleftarrow$ tune ramp direction
$\alpha < 0$, islands coming inwards
no particle trapped
but emittance increases

$\longleftarrow$ tune ramp direction
$\alpha > 0$, islands going outwards
with increasing size
particles trapped
emittance increases w/o limit

A Ring of Particles, $\alpha > 0$

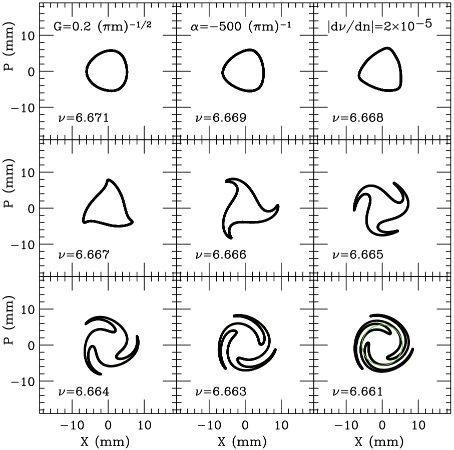


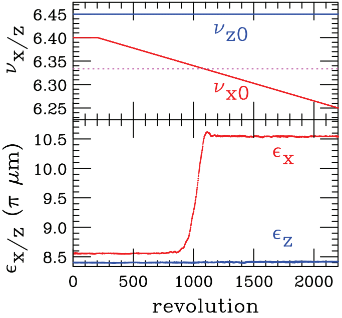


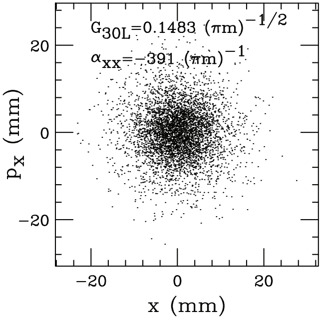


- Green: original ring beam.
- Right plots: for a Gaussian bunch.
- ϵ_x increases without limit.

A Ring of Particles, $\alpha < 0$

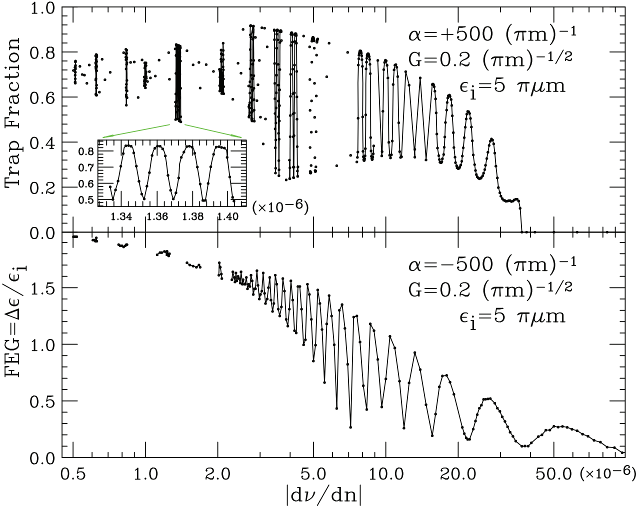






- Green: original ring beam.
- Right plots: for a Gaussian bunch.
- Emittance increase is limited.

A Ring of Particles



- For $\alpha > 0$, particles trapped in islands.
- Study *trap fraction*
- Oscillations do not converge as rapidly as when $\alpha < 0$

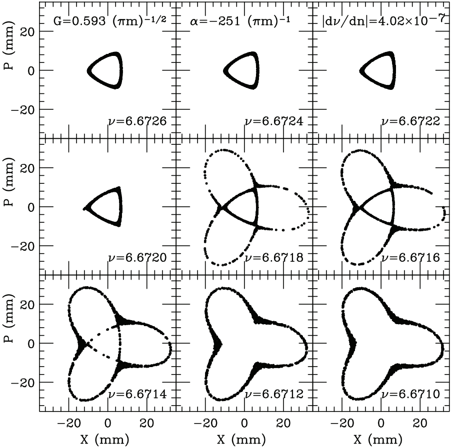
- For $\alpha < 0$, talk about *fractional emittance growth* FEG$=\dfrac{\Delta\epsilon}{\epsilon_i}$.

- Note large oscillations in trap fraction and FEG due to initial timing.
- oscillation amplitudes are small only when tune ramp rates are small.

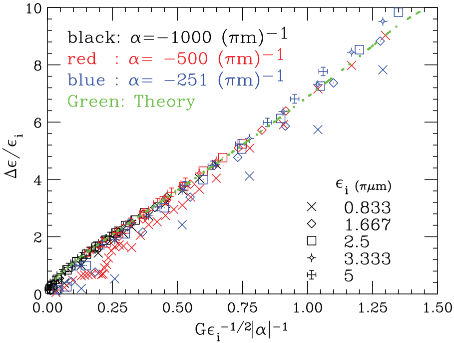
Ring of Particles with Adiabatic Tune Ramp

- Phase space evolution for $\alpha < 0$ and *adiabatic tune ramp* ($|d\nu/dn| \lesssim 4.02 \times 10^{-7}$ per turn)
- Fractional emittance growth (FEG) scales:

$$\frac{\Delta\epsilon}{\epsilon_i} \approx 7.3 \frac{G}{\epsilon_i^{1/2}|\alpha|}$$



- FEG $= \dfrac{outer\ island\ area}{inner\ island\ area}$, and is detuning α dependent.

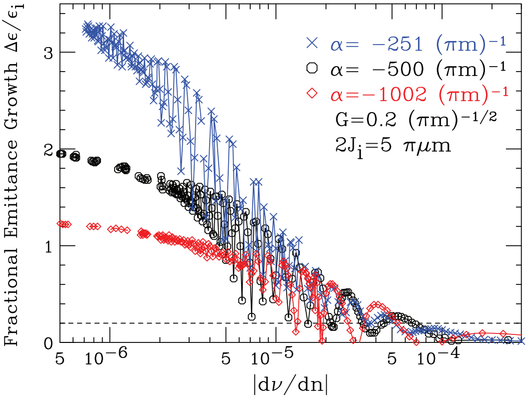


Scaling Law of FEG at Adiabatic Ramping with $\alpha < 0$

- For a beam, $\mathrm{FEG} = 7.3 \int \frac{G\sqrt{J}}{|\alpha|\epsilon_i}\rho(J)dJ \xrightarrow{\text{Gaussian}} 7.3\Gamma\left(\frac{3}{2}\right)\frac{G}{\underbrace{\sqrt{\epsilon_i}|\alpha|}}$.

  ↑ *adiabatic scaling parameter*

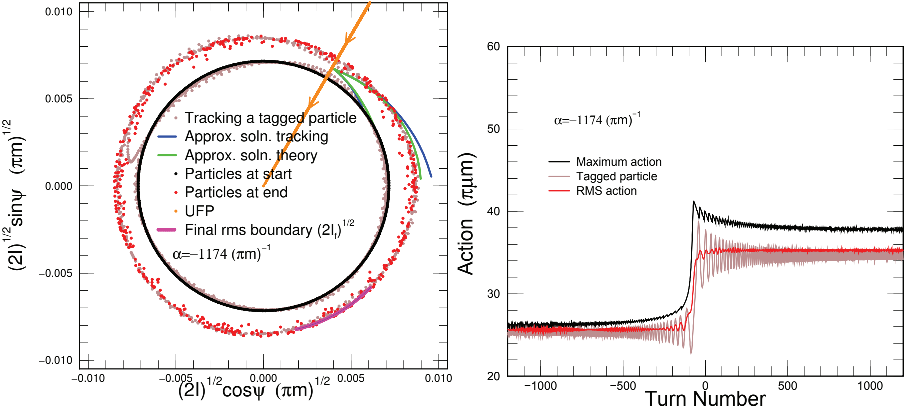
- Note that FEG $\propto \alpha^{-1}$ because particles have time to follow separatrices of islands.



- In an FFAG, tune ramp rate is *not* adiabatic.
- Emittance growth should be independent of detuning as shown in simulation.
- Study of detuning dependency is an aim of this talk.

Emittance Growth after Encountering an UFP $\alpha < 0$

- Let us see how emittance increases when passing through an UFP. Start with a ring of particles representing outermost of beam.



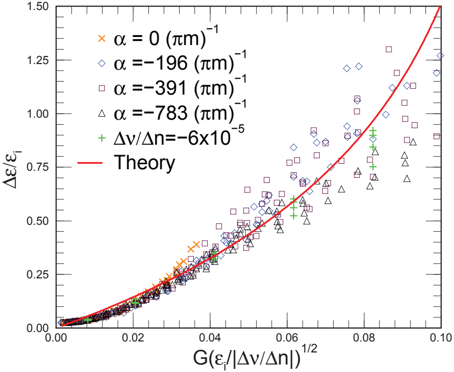
- Green: A particle collides with u.f.p. and increases in action (theory).
- Brown: A particle nearly collides with an UFP at $\psi = \pi$
- Near an unstable fixed point, ΔI related to $\Delta\psi$.

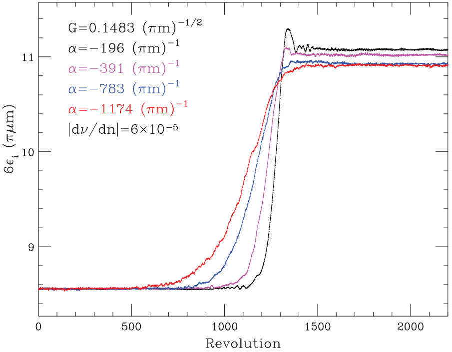
$$\Delta\psi \approx \frac{\pi}{10}$$

Emittance Growth After Encountering UFP $\alpha < 0$

- Hamilton's equations of motion:
  $I' = 3GI^{3/2}\sin 3\psi, \quad \psi' = \delta + \alpha I + \frac{3}{2}GI^{1/2}\cos 3\psi$
- Expand about an UFP where $\psi' = 0, \quad I' = 0$
- With $\psi'' = \delta' = \dfrac{1}{2\pi}\dfrac{d\nu}{dn}$, $\begin{cases} \Delta I \approx 6\pi^2 G I_u^{3/2}\dfrac{d\nu}{dn}(n-n_u)^3 \\ \Delta\psi \approx \pi\dfrac{d\nu}{dn}(n-n_u)^2 \end{cases}$
- Eliminating $(n-n_u)$, $\quad \Delta I \approx \dfrac{6\sqrt{\pi}GI_u^{3/2}}{\left|\frac{d\nu}{dn}\right|}(\Delta\psi)^{3/2}$
- Roughly $I_u = \dfrac{1}{2}(I_i + I_f) = I_i\left(1 + \dfrac{\Delta I}{2I_i}\right)$ with $\Delta I = I_f - I_i$
- With $I_i = 3I\epsilon_i$, $\boxed{\dfrac{\Delta\epsilon}{\epsilon_i} = \dfrac{12\sqrt{3\pi}G\epsilon_i^{1/2}(\Delta\psi)^{3/2}}{\sqrt{\left|\frac{d\nu}{dn}\right|}}\left(1 + \dfrac{\Delta\epsilon}{2\epsilon_i}\right)^{3/2}}$
- Scaling parameter: $\boxed{S = G\sqrt{\dfrac{\epsilon_i}{|d\nu/dn|}}}$ independent of detuning α.

Comparison with Simulations





- **Simulation inputs:**
  α from 0 to -800 $(\pi\text{m})^{-1}$,
  G from 0.02 to 0.8 $(\pi\text{m})^{-\frac{1}{2}}$,
  $|d\nu/dn|$ from 10^{-5} to 10^{-2},
  ϵ_i : 0.93, 2.3, 4.62, 6.94 $(\pi\mu\text{m})$

Conclusions:

- Scaling law fits simulation results pretty well.
- Emittance growth across resonance is almost detuning α independent.

Comments

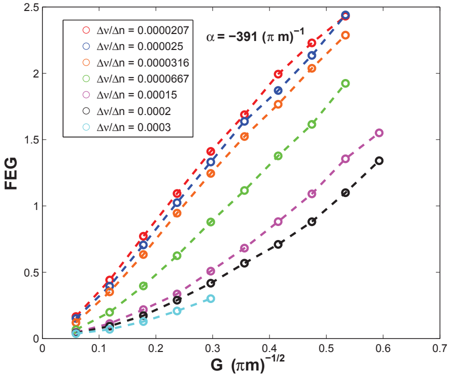
- Is Taylor expansion valid?
- $2\pi(n - n_u)$ appears to be a big number when expand about UFP
- Looking into more terms in expansion:

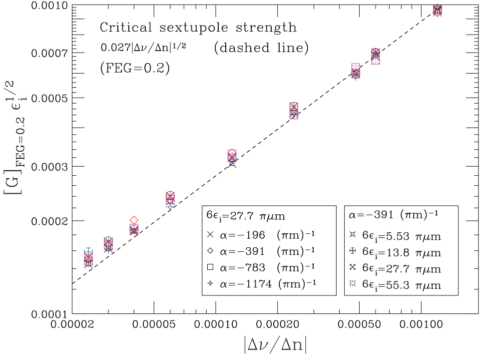
  expansion is actually in

  $\left[GI_u^{1/2} 2\pi(n - n_u)\right]^2 \approx 12\pi S^2 \Delta\psi$
- When $S = G\sqrt{\frac{\epsilon_i}{|d\nu/dn|}} = 0.1$ with $I_u \sim 3\epsilon_i$

  $\left[GI_u^{1/2} 2\pi(n - n_u)\right]^2 \approx 12\pi S^2 \Delta\psi \approx 0.12$
- Expansion breaks down for adiabatic tune rates, $\because$ S will be large.
- For adiabatic tune ramp, solve Hamiltonian without perturbation.

Scaling Law for Critical Resonance Strength $\alpha < 0$





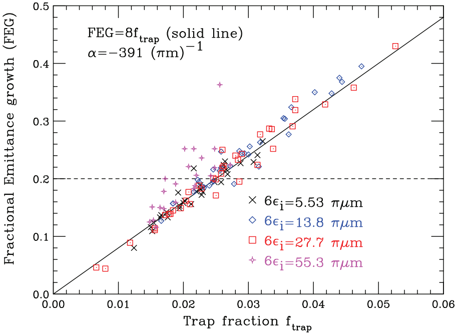
- FEG vs G with $\alpha = -391\ (\pi\mathrm{m})^{-1}$
- Read off tolerable or critical resonance strength $[G]_{\mathrm{FEG}=0.2}$

- $[G]_{\mathrm{FEG}=0.2} = 0.027\epsilon_i^{-1/2}\left|\frac{\Delta\nu}{\Delta n}\right|^{1/2}$
- Factor 0.027 agrees with derived scaling law when FEG$=0.2$ and initial slope $F = 6.48$.

Trap Fraction vs Emittance Growth Factor (EGF)

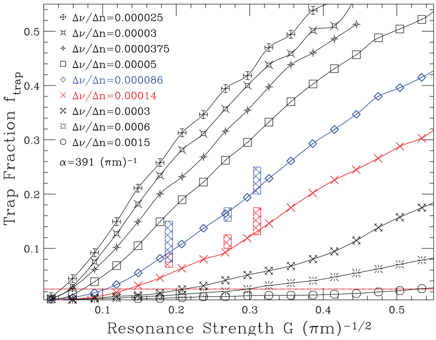
- For $\alpha < 0$, define *fractional emittance growth* FEG
- For $\alpha > 0$, define $f_{\rm trap} = \dfrac{N_{J>J_{i,max}}}{N_{\rm total}}$ more than trapping!!
- $f_{\rm trap}$, as defined, can also be used when $\alpha < 0$.

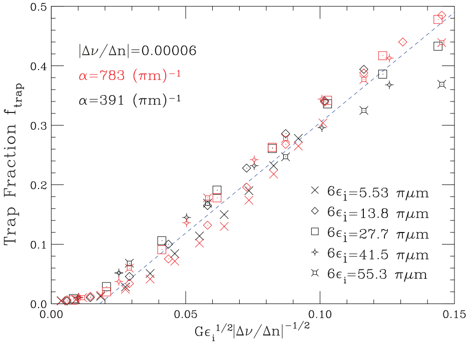
  Then there must be a relation between FEG and $f_{\rm trap}$.



- Observe a linear relation between $f_{\rm trap}$ and FEG.
- If we define a *tolerable emittance growth* as 20% $(\mathrm{FEG}{=}0.2)$, corr. *trap fraction* is $f_{\rm trap} = 2.5\%$.

Trap Fraction f_{trap} at Positive Detuning ($\alpha > 0$)





- $\epsilon_i = 4.62\,(\pi\mu\text{m})$
  $\alpha = 391\ (\pi\text{m})^{-1}$
- Simulations agree with KEK experimental results.
- Experimental results: boxes

- Linear relation: f_{trap} vs S $\left(\begin{array}{c}\textit{scaling}\\ \textit{parameter}\end{array}\right)$
- Data less clustered than when $\alpha < 0$
- Worst when include more data of larger detunings and ramp rates.

Critical Resonance Strength for $\alpha > 0$

- Scaling law shows a small dependency on detuning α.
- As α increases, power in $|d\nu/dn|$ increases from $\frac{1}{2}$ to $\frac{2}{3}$.

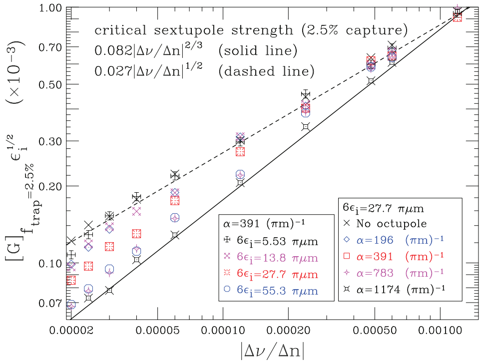
- Large α in solid line:

$$[G]_{f_{\rm trap}=2.5\%} = 0.082\epsilon_i^{-1/2} \left|\frac{\Delta\nu}{\Delta n}\right|^{\frac{2}{3}}$$

- Small $\alpha > 0$ in dashes:

$$[G]_{f_{\rm trap}=2.5\%} = 0.027\epsilon_i^{-1/2} \left|\frac{\Delta\nu}{\Delta n}\right|^{\frac{1}{2}}$$

- Above is exactly the same as scaling law for $[G]_{\rm FEG=0.2}$ when $\alpha < 0$



- Larger $\alpha \to$ larger tune spread $\to$ longer resonance crossing time $\to$ more particle trapped in islands.
- Larger α, smaller islands, less trapping.

Comparison with Aiba's Result

- Ours: $\begin{cases}\text{tolerable}\\ \text{resonance}\\ \text{strength}\end{cases}$ $\boxed{[G]_{f_{\rm trap}=2.5\%} = A\epsilon_i^{-1/2}\left|\frac{\Delta\nu}{\Delta n}\right|^p}$ $\begin{cases}\text{scaling}\\ \text{parameter}\end{cases}$ $\boxed{S = \frac{G\epsilon^{1/2}}{|d\nu/dn|^{1/2}}}$

  with A varying from 0.027 to 0.082 and p from $\frac{1}{2}$ to $\frac{2}{3}$.

- Aiba's: $\begin{cases}\text{trap}\\ \text{efficiency}\end{cases}$ $\boxed{P_T = \frac{\pi}{2^{1/2}}\left(\frac{G}{3^{1/3}|\alpha|\epsilon_i^{1/2}}\right)^{\frac{1}{2}}\eta^{-\frac{1}{4}}e^{-\eta_{\rm ad}}}$

  with $\eta_{\rm ad} = \left[\frac{|d\nu/dn|}{3^{1/2}36|\alpha|G\epsilon_i^{3/2}}\right]^{2/3}$ and $\eta = \begin{cases}\eta_{\rm ad} & \eta_{\rm ad} > 1\\ 1 & \eta_{\rm ad} < 1\end{cases}$ [0.10in]

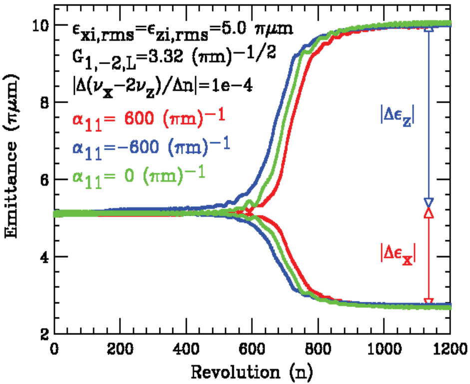
- The two results are very different in
  - dependency in detuning α
  - dependency in initial emittance ϵ_i
  - dependency in tune-ramp rate $d\nu/dn$
  - different scaling parameter

- According to Aiba etal, trapping will be greatly reduced by increasing detuning while crossing resonance. But our result predicts not much help.

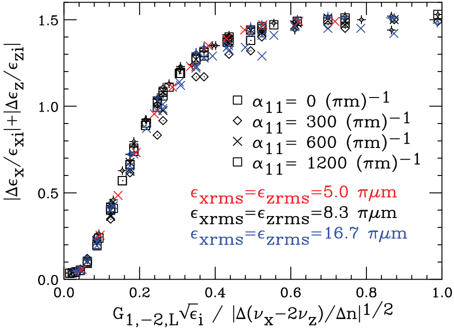
3rd-Order Difference Resonance $\nu_x - 2\nu_z = \ell$

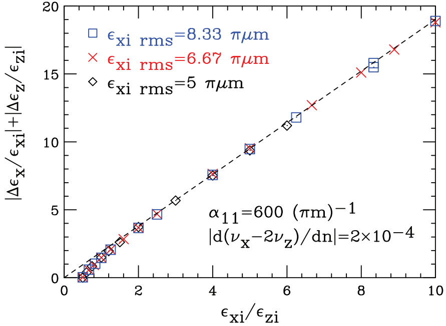
- This resonance $\nu_x - 2\nu_z = \ell$ is known as *Walkinshaw resonance* in the cyclotron circle.
- Not easy to avoid in cyclotrons, since $\nu_x \sim \gamma$ and $\nu_z < 1$.
- Near or crossing this resonance leads to emittance increase.
- Not catastrophic and emittance increase is limited.
- This is because $2J_x + J_z = J_2$, which is a constant of motion.
- $\therefore\ 2\Delta\epsilon_x + \Delta\epsilon_z = 0$

  ϵ_x shrinks and ϵ_z grows
- Note that emittance growths crossing Walkinshaw resonance are independent of detuning $\alpha_{11} = \alpha_{xx} + 4\alpha_{zz} - 4\alpha_{xz}$.



Scaling Laws





- There is a scaling law with the scale parameter

$$S = G_{1-2\ell}\left[\frac{\epsilon_i}{|d(\nu_x-2\nu_z)/dn|}\right]^{1/2} \qquad \text{when } \epsilon_{xi} = \epsilon_{zi}\ .$$

- The growth rates become linear with $\epsilon_{xi}/\epsilon_{zi}$ when latter ratio is large.
- There is no growth when $\epsilon_{xi}/\epsilon_{zi} \lesssim \frac{1}{2}$.
- To avoid Walkinshaw resonance may be should operate with $\epsilon_{xi} < \epsilon_{zi}$.

Conclusions

- We study emittance growth crossing 3rd-order resonance and derive some formulas using simulation and theoretical derivation, giving pretty good agreement with experimental measurements.
- With *tolerable FEG of 20%* or *trap fraction of 2.5%*, we derive scaling laws for critical resonance strength $[G]_{\rm FEG=0.2}$ and $[G]_{f_{\rm trap}=0.2}$.
- Perturbation about UFP is a good method in deriving scaling law. Can be applied to FEG for crossing other nonlinear resonance. For example, for octupole driven resonances, we obtain FEG scales with $G_{40\ell}\epsilon_i|d\nu/dn|^{-1/2}$, independent of detuning.
- Our scaling law can be useful in design of high power accelerators, estimate of emittance growth in cyclotron, and estimate of requirement of slow beam extraction using 3rd-order resonance.
- Our scaling law is very different from the one derived by Aiba etal. A non-scaling FFAG has recently commissioned. Our scaling law should be timely for experimental test.