Scaling Properties of Resonances in Non-Scaling FFAGs

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Introduction

- During ramping of an FFAG, betatron tunes cross many nonlinear resonances.

- We study here emittance growth and beam loss crossing the 3rd-order resonance $3\nu_x = \ell$.

- Chao et al and Aiba et al attempted to derive trap efficiency during resonance crossing. There are successes, but do not fit experimental results well.

- We set 20% as tolerable emittance increase or 2.5% as tolerable trap-fraction in resonance crossing, and derive scaling laws for critical allowable resonance strength.

- The scaling law can be obtained by solving Hamilton’s equations of motion by perturbation.

- Will comment on crossing $\nu_x - 2\nu_z = \ell$ resonance.
The model ring for simulation is similar to the Fermilab Booster: 
$C = 474$ m with 24-fold symmetric FODO lattice.

Betatron functions are:
$\beta_{xF} = 40.0$ m, $\beta_{zF} = 8.3$ m
$\beta_{xD} = 6.3$ m, $\beta_{zD} = 21.4$ m

A sextupole and octupole at one of the D-quads provide 3rd-order nonlinearity and detuning.

Kinetic beam energy is fixed at 1 GeV.

The simulations ramp the horizontal tune:
$\nu_x$: from 6.4 to 6.24 crossing $3\nu_x = 19$ resonance
$\nu_z$: fixed at 6.45, but chosen not to excite difference or sum resonance as well as 4th-order resonances.
**Fixed Points**

- Hamiltonian in the horizontal phase space:

\[ H = \delta I + \frac{1}{2} \alpha I^2 + GI^{3/2} \cos 3\psi, \]  
proximity \( \delta = \nu - \frac{\ell}{3} \), detuning \( \alpha \).

- We study *downward ramping* of horizontal tune \( \nu \).

\[ \begin{array}{c}
\text{\( \alpha < 0 \)} \\
\text{\( \alpha > 0 \)}
\end{array} \]

\[ \left( \frac{I_{fp}\alpha}{I^2} \right)^2/G \]

\( |\alpha|\delta/G^2 \)

- \( \leftarrow \) tune ramp direction
- \( \alpha < 0 \), islands coming inwards
- no particle trapped
- but emittance increases

- \( \leftarrow \) tune ramp direction
- \( \alpha > 0 \), islands going outwards
- with increasing size
- particles trapped
- emittance increases w/o limit

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A Ring of Particles, $\alpha > 0$

- **Green**: original ring beam.
- **Right plots**: for a Gaussian bunch.
- $\epsilon_x$ increases without limit.

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A Ring of Particles, $\alpha < 0$

- **Green**: original ring beam.
- **Right plots**: for a Gaussian bunch.
- **Emittance increase is limited.**
For $\alpha > 0$, particles trapped in islands.

- Study *trap fraction*

- Oscillations do not converge as rapidly as when $\alpha < 0$

For $\alpha < 0$, talk about *fractional emittance growth* $\text{FEG} = \frac{\Delta \epsilon}{\epsilon_i}$.

- Note large oscillations in *trap fraction* and FEG due to initial timing.

- Oscillation amplitudes are small only when tune ramp rates are small.
Ring of Particles with Adiabatic Tune Ramp

- Phase space evolution for $\alpha < 0$ and *adiabatic tune ramp* ($|d\nu/dn| \lesssim 4.02 \times 10^{-7}$ per turn)
- Fractional emittance growth (FEG) scales:
  
  \[
  \frac{\Delta \epsilon}{\epsilon_i} \approx 7.3 \frac{G}{\epsilon_i^{1/2} |\alpha|}
  \]

- **FEG** = \frac{\text{outer island area}}{\text{inner island area}},
  and is detuning $\alpha$ dependent.

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Scaling Law of FEG at Adiabatic Ramping with $\alpha < 0$

- For a beam, $\text{FEG} = 7.3 \int \frac{G \sqrt{J}}{|\alpha| \epsilon_i} \rho(J) dJ \xrightarrow{\text{Gaussian}} 7.3 \Gamma \left( \frac{3}{2} \right) \frac{G}{\sqrt{\epsilon_i |\alpha|}}$.

- Note that $\text{FEG} \propto \alpha^{-1}$ because particles have time to follow separatrices of islands.

- In an FFAG, tune ramp rate is \textit{not} adiabatic.

- Emittance growth should be independent of detuning as shown in simulation.

- Study of detuning dependency is an aim of this talk.
Emittance Growth after Encountering an UFP $\alpha < 0$

- Let us see how emittance increases when passing through an UFP. Start with a ring of particles representing outermost of beam.

- **Green**: A particle collides with u.f.p. and increases in action (theory).
- **Brown**: A particle nearly collides with an UFP at $\psi = \pi$

Near an unstable fixed point, $\Delta l$ related to $\Delta \psi$. $\Delta \psi \approx \frac{\pi}{10}$
Emittance Growth After Encountering UFP $\alpha < 0$

- Hamilton’s equations of motion:
  \[ l' = 3GI^{3/2} \sin 3\psi, \quad \psi' = \delta + \alpha l + \frac{3}{2} GI^{1/2} \cos 3\psi \]

- Expand about an UFP where $\psi' = 0, \quad l' = 0$

- With $\psi'' = \delta' = \frac{1}{2\pi} \frac{d\nu}{dn}$, \[
\begin{align*}
\Delta l &\approx 6\pi^2 GI_u^{3/2} \frac{d\nu}{dn} (n-n_u)^3 \\
\Delta \psi &\approx \pi \frac{d\nu}{dn} (n-n_u)^2
\end{align*}
\]

- Eliminating $(n-n_u)$,
  \[ \Delta l \approx \frac{6\sqrt{\pi} GI_u^{3/2}}{\left| \frac{d\nu}{dn} \right|} (\Delta \psi)^{3/2} \]

- Roughly $l_u = \frac{1}{2}(l_i + l_f) = l_i \left(1 + \frac{\Delta l}{2l_i}\right)$ with $\Delta l = l_f - l_i$

- With $l_i = 3l_i\epsilon_i$,
  \[ \frac{\Delta \epsilon}{\epsilon_i} = \frac{12\sqrt{3\pi} G\epsilon_i^{1/2} (\Delta \psi)^{3/2}}{\sqrt{|\frac{d\nu}{dn}|}} \left(1 + \frac{\Delta \epsilon}{2\epsilon_i}\right)^{3/2} \]

- Scaling parameter:
  \[ S = G \sqrt{\frac{\epsilon_i}{|\frac{d\nu}{dn}|}} \] independent of detuning $\alpha$.  

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Comparison with Simulations

- **Simulation inputs:**
  - $\alpha$ from 0 to $-800 \ (\pi \text{m})^{-1}$,
  - $G$ from 0.02 to 0.8 $\ (\pi \text{m})^{-1/2}$,
  - $|d\nu/dn|$ from $10^{-5}$ to $10^{-2}$,
  - $\epsilon_i : 0.93, 2.3, 4.62, 6.94 \ (\pi \mu \text{m})$

Conclusions:
- **Scaling law** fits simulation results pretty well.
- Emittance growth across resonance is almost *detuning* $\alpha$ independent.
Is Taylor expansion valid?

$2\pi(n - n_u)$ appears to be a big number when expand about UFP

Looking into more terms in expansion:

expansion is actually in

$$\left[ Gl_u^{1/2} 2\pi(n - n_u) \right]^2 \approx 12\pi S^2 \Delta \psi$$

When $S = G \sqrt{\frac{\epsilon_i}{|d\nu/dn|}} = 0.1$ with $l_u \sim 3\epsilon_i$

$$\left[ Gl_u^{1/2} 2\pi(n - n_u) \right]^2 \approx 12\pi S^2 \Delta \psi \approx 0.12$$

Expansion breaks down for adiabatic tune rates, $\therefore S$ will be large.

For adiabatic tune ramp, solve Hamiltonian without perturbation.
Scaling Law for Critical Resonance Strength $\alpha < 0$

- FEG vs $G$ with $\alpha = -391 \ (\pi m)^{-1}$
- Read off tolerable or critical resonance strength $[G]_{\text{FEG}=0.2}$

$[G]_{\text{FEG}=0.2} = 0.027 \epsilon^{-1/2} |\Delta \nu/\Delta n|^{1/2}$

Factor 0.027 agrees with derived scaling law when FEG = 0.2 and initial slope $F = 6.48$. 

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For $\alpha < 0$, define \textit{fractional emittance growth} FEG

For $\alpha > 0$, define $f_{\text{trap}} = \frac{N_{J>J_{i,\text{max}}}}{N_{\text{total}}}$ more than trapping!!

$f_{\text{trap}}$, as defined, can also be used when $\alpha < 0$.

Then there must be a relation between FEG and $f_{\text{trap}}$.

Observe a linear relation between $f_{\text{trap}}$ and FEG.

If we define a \textit{tolerable emittance growth} as 20% (FEG=0.2),
corr. \textit{trap fraction} is $f_{\text{trap}} = 2.5\%$. 
Trap Fraction $f_{\text{trap}}$ at Positive Detuning ($\alpha > 0$)

\[ \epsilon_i = 4.62 \, (\pi \mu m) \]
\[ \alpha = 391 \, (\pi m)^{-1} \]

- Simulations agree with KEK experimental results.
- Experimental results: boxes
- Data less clustered than when $\alpha < 0$
- Worst when include more data of larger detunings and ramp rates.
Critical Resonance Strength for $\alpha > 0$

- Scaling law shows a small dependency on detuning $\alpha$.
- As $\alpha$ increases, power in $|d\nu/dn|$ increases from $\frac{1}{2}$ to $\frac{2}{3}$.

Large $\alpha$ in solid line:

$$[G]_{f_{\text{trap}}=2.5\%} = 0.082\epsilon_i^{-1/2} \left(\frac{\Delta\nu}{\Delta n}\right)^{2/3}$$

Small $\alpha > 0$ in dashes:

$$[G]_{f_{\text{trap}}=2.5\%} = 0.027\epsilon_i^{-1/2} \left(\frac{\Delta\nu}{\Delta n}\right)^{1/2}$$

Above is exactly the same as scaling law for $[G]_{\text{FEG}=0.2}$ when $\alpha < 0$

- Larger $\alpha \rightarrow$ larger tune spread $\rightarrow$ longer resonance crossing time $\rightarrow$ more particle trapped in islands.
- Larger $\alpha$, smaller islands, less trapping.
Comparison with Aiba’s Result

- **Ours:**
  \[
  [G]_{\text{trap}} = 2.5\% = A \epsilon_i^{-1/2} \left| \frac{\Delta \nu}{\Delta n} \right|^p
  \]
  with \(A\) varying from 0.027 to 0.082 and \(p\) from \(\frac{1}{2}\) to \(\frac{2}{3}\).

- **Aiba’s:**
  \[
  P_T = \frac{\pi}{2^{1/2}} \left( \frac{G}{3^{1/3} |\alpha| \epsilon_i^{1/2}} \right)^{1/2} \eta^{-1/4} e^{-\eta_{ad}}
  \]
  with \(\eta_{ad} = \left[ \frac{|d\nu/dn|}{3^{1/2} 36 |\alpha| G \epsilon_i^{3/2}} \right]^{2/3}\) and \(\eta = \begin{cases} \eta_{ad} & \eta_{ad} > 1 \\ 1 & \eta_{ad} < 1 \end{cases}\)

- The two results are very different in
  - dependency in detuning \(\alpha\)
  - dependency in initial emittance \(\epsilon_i\)
  - dependency in tune-ramp rate \(d\nu/dn\)
  - different scaling parameter

According to Aiba et al, trapping will be greatly reduced by increasing detuning while crossing resonance. But our result predicts not much help.
3rd-Order Difference Resonance $\nu_x - 2\nu_z = \ell$

- This resonance $\nu_x - 2\nu_z = \ell$ is known as *Walkinshaw resonance* in the cyclotron circle.
- Not easy to avoid in cyclotrons, since $\nu_x \sim \gamma$ and $\nu_z < 1$.
- Near or crossing this resonance leads to emittance increase.
- Not catastrophic and emittance increase is limited.
- This is because
  
  $2J_x + J_z = J_2,$

  which is a constant of motion.

- $2\Delta \epsilon_x + \Delta \epsilon_z = 0$

  $\epsilon_x$ shrinks and $\epsilon_z$ grows

- Note that emittance growths crossing Walkinshaw resonance are independent of detuning
  
  $\alpha_{11} = \alpha_{xx} + 4\alpha_{zz} - 4\alpha_{xz}$.
There is a scaling law with the scale parameter

\[ S = G_{1-2\ell} \left[ \frac{\epsilon_i}{|d(\nu_x-2\nu_z)/dn|} \right]^{1/2} \]

when \( \epsilon_{xi} = \epsilon_{zi} \).

The growth rates become linear with \( \epsilon_{xi}/\epsilon_{zi} \) when latter ratio is large.

There is no growth when \( \epsilon_{xi}/\epsilon_{zi} \lesssim \frac{1}{2} \).

To avoid Walkinshaw resonance may be should operate with \( \epsilon_{xi} < \epsilon_{zi} \).
Conclusions

- We study emittance growth crossing 3rd-order resonance and derive some formulas using simulation and theoretical derivation, giving pretty good agreement with experimental measurements.

- With *tolerable FEG of 20%* or *trap fraction of 2.5%*, we derive scaling laws for critical resonance strength \([G]_{\text{FEG}=0.2}\) and \([G]_{f\text{trap}=0.2}\).

- Perturbation about UFP is a good method in deriving scaling law. Can be applied to FEG for crossing other nonlinear resonance. For example, for octupole driven resonances, we obtain FEG scales with \(G_{40\ell\epsilon_i}|d\nu/dn|^{-1/2}\), independent of detuning.

- Our scaling law can be useful in design of high power accelerators, estimate of emittance growth in cyclotron, and estimate of requirement of slow beam extraction using 3rd-order resonance.

- Our scaling law is very different from the one derived by Aiba et al. A non-scaling FFAG has recently commissioned. Our scaling law should be timely for experimental test.