Equipartition, Reality or Swindle ?

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-1- The EQP theorem, validity limit

-2- EQP and linac beam dynamics

-3- Where is the swindle ?

~~(-4- Why the formula presently used to define EQP is wrong ?)~~

-5- Why energy exchanges can occur although the EQP rule is respected ?

-6- Why safe tunings can be found although the EQP rule is not respected ?

-7- Why LINAC designers nevertheless like to use the EQP rule ?

DISCUSSION

The EQP theorem

A fundamental law in classical statistical mechanics

" The total energy of a system in thermal equilibrium
is shared equally amongst all its
energetically accessible independent degrees of freedom "

Ex : Ideal mono-atomic gas with N "particles" confined in a box

EQP Theorem => The average kinetic energies
in every one of the 3N translational degrees of freedom
shall be equal when the system will be in equilibrium

$$\frac{1}{T}\int_0^T v_{x1}^2\,dt = \frac{1}{T}\int_0^T v_{y1}^2\,dt = \frac{1}{T}\int_0^T v_{z1}^2\,dt = \ldots = \frac{1}{T}\int_0^T v_{xN}^2\,dt = \frac{1}{T}\int_0^T v_{yN}^2\,dt = \frac{1}{T}\int_0^T v_{zN}^2\,dt$$

$T \to \infty$

**The EQP theorem concerns the 3N kinetic energies
averaged over time of each one of the N particles**

… Ex : N "particles" confined in a box

Explanation : At the microscopic level, the energy transfer induced by the collisions between the particles has an equal probability to be done towards all the 3N degrees of freedom

Large N systems

Equal 3N mean kinetic energies
averaged over time of each one of the N particles
<=>
Equal mean kinetic energies in the x, y and z translational degrees of freedom,
the averaging being done over the N particles at a given time

$$\frac{1}{N}\sum_{i=1}^{N} v_{xi}^2 = \frac{1}{N}\sum_{i=1}^{N} v_{yi}^2 = \frac{1}{N}\sum_{i=1}^{N} v_{zi}^2 \quad \text{or} \quad \langle v_x^2 \rangle = \langle v_y^2 \rangle = \langle v_z^2 \rangle$$

Description of the macroscopic system behaviour

A consequence of the previous equality which describes the microscopic behaviour of the systems when the EQP theorem apply

The law of equipartition holds only for ergodic systems in thermal equilibrium

A Hamiltonian system with n degrees of freedom
is ergodic if
its state represented by a point in the 2n-phase space (q_1, ..., q_n, p_1, ..., p_n)
will pass equally often on every point of the constant-energy surface
in this 2n-phase space during its long term evolution

.

A system is ergodic when the energy surface cannot be divided into finite regions such that, if the initial point in phase-space is located in one such region, the system trajectory remains entirely within that region
(John von Neumann, 1932)

In an ergodic system, the position and energy of each particle must be spread all over the constant energy surface

Ergodicity in nonlinear Hamiltonian syst.

The complexity of the phase-space trajectories evolves with the nonlinearity level (weak / strong nonlinearity) and with the perturbing forces strength (K) :

- **Complete integrability** without perturbation (K = 0) :
  Phase-space trajectories of the resonant particles represented by fix points
  Non-resonant trajectories by continuous lines

- **KAM integrability** for a weak perturbation (K → 0, weakly non-integrable Hamiltonian system) :
  Most of the non-resonant trajectories slightly deformed but remains continuous,
  they form "KAM impassable barriers" which limit the accessible domain for the other particles
  The separatrix associated to the resonances are destroyed, narrow chaotic layers appears

- **Complete chaos** reached when the perturbation is increased (K → ∞) :
  The particle motions are chaotic everywhere in phase space

Nonlinear Hamiltonian systems become ergodic only when the perturbations are strong enough to lead to the destruction of the KAM barriers nearly everywhere in phase space

This is the condition which authorizes the application of the EQP theorem

Equipartition and linac beam dynamics

"Linac beam EQP rule" formulated assuming that $\langle v_x^2 \rangle = \langle v_y^2 \rangle = \langle v_z^2 \rangle$

2 identical transverse motions =>

Equality between the rms energy spread in one transverse phase-plane

and the rms energy spread in the longitudinal phase-plane : $E_{x\_rms} = E_{z\_rms}$

(Confusion between mean values (as stated by the EQP theorem) and rms values !)

EQP formula in terms of emittances, beam sizes and phase advances

$$d^2x_{rms}/dt^2 + \omega_x^2 x_{rms} = 0$$

$$\rightarrow \; a' = \omega_x a \quad \rightarrow \; E_{x\_rms} = 1/2\, m\, a'^2 = 1/2\, m\, \omega_x^2 a^2$$

$$\rightarrow \; \varepsilon_x = \pi\, a\, a' = \omega_x a^2$$

+ identical rms linear equations in the longitudinal phase plane

$$\frac{a}{b} = \frac{\varepsilon_x}{\varepsilon_z} = \frac{\sigma_z}{\sigma_x}$$

EQP rule

(Phase advances per unit of time ω replaced by the phase advances per lattice σ)

Where is the swindle ?

The main equipartitionist' mistake comes from the fact that they apply the EQP theorem to systems which are obviously and hopefully not ergodic

EQP rule formulated even though nobody demonstrates, even discusses, the applicability of the EQP theorem !

Linac designs => safe working points with quite smooth and regular particle trajectories, even with severe tune depressions (sometimes up to 0.5), even when the EQP rule is not respected !

Realistic linac designs lead to "weak perturbation regimes" with phase-spaces mainly inhabited with slightly deformed non-resonant trajectories and chaotic trajectories in limited and confined areas

The second equipartitionist' swindle comes from the fact that they promote the use the EQP rule to avoid energy exchanges and halo formation

This is wrong !

♠ Energy exchanges can happen although the EQP rule is respected

♥ Safe working points can be found more easily when the EQP rule is not respected

Energy exchange in EQP beam

Why energy exchanges can occur although the EQP rule is respected ?

Because
the EQP rule impose a working point with s-c located on the line

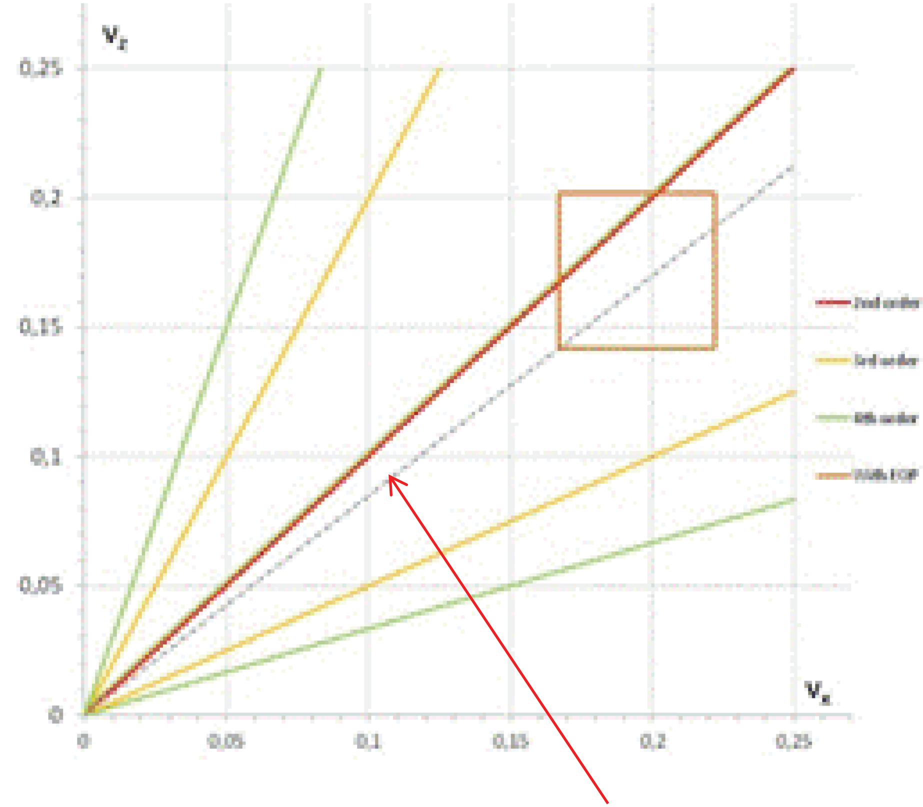
$$\mathbf{v_z = (\varepsilon_x / \varepsilon_z)\, v_x}$$

what is not a means to avoid the coupling resonances

$$\mathbf{n_1 v_x + n_2 v_z = p}$$

A coupling resonance can affect the beam core when its slope is close to the slope of the EQP line (the emittance ratio $\varepsilon_x / \varepsilon_z$)

The EQP rule is not a mean to avoid energy / emittance exchanges !



EQP => $\varepsilon_x / \varepsilon_z = \sigma_z / \sigma_x = 0.85$

$\sigma_x / \sigma_{0x} = 0.75$ $\sigma_z / \sigma_{0z} = 0.70$

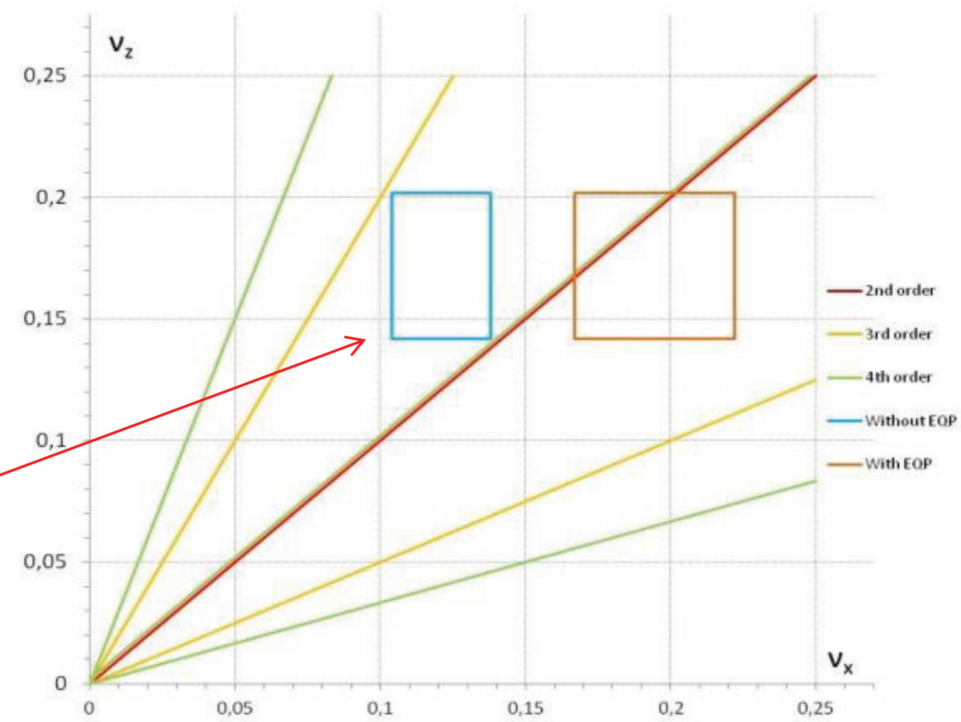
$\sigma_{0x} = 80°$ ($v_{0x} = 0.222$) $\sigma_x = 60°$ ($v_x = 0.167$)

$\sigma_{0z} = 73°$ ($v_{0z} = 0.202$) $\sigma_z = 51°$ ($v_z = 0.142$)

Safe working point with non EQP beam

Why safe working points can be found more easily when the EQP rule is not respected ?

Because
without the strong constraint of the EQP rule which dictates the choice of one tune when the other is fixed,
the **free choice of both radial and longitudinal tunes** allows to select a working point which seats the beam footprint out of the coupling resonances



The application of the EQP rule can lead to a non-optimal beam dynamics and/or higher accelerator construction and operation costs

$\varepsilon_x / \varepsilon_z = 0.85$ $\sigma_x / \sigma_{0x} = 0.75$ $\sigma_z / \sigma_{0z} = 0.70$

$\boldsymbol{\sigma_{0x} = 50° \ (\nu_{0x} = 0.139)}$ $\boldsymbol{\sigma_x = 36° \ (\nu_x = 0.104)}$

$\sigma_{0z} = 73° \ (\nu_{0z} = 0.202)$ $\sigma_z = 51° \ (\nu_z = 0.142)$

$$\boldsymbol{\sigma_x \varepsilon_x / \varepsilon_z \sigma_z = 0.62} \quad \neq 1 \ (\text{EQP})$$

Why EQP in the linac microcosm ?

"Human beings like to believe in simple to understand and simple to put in practice ideas"

To consider the effect of the coupling resonances, to evaluate their level of excitation, to avoid them when they have significant bad effects… is more complex !

Analogy with the "Pascal's gambit" :

"If I believe in God and there is no God then I have lost nothing, however if I don't believe in God and there is a God then I will go to hell, therefore it is rational to believe in God"

The transposition to the belief in EQP is a serious mistake

The application of the EQP rule can lead to a non-optimal beam dynamics and/or higher accelerator construction and operation costs, especially when the EQP rule implies the choice of RF systems and accelerating cavities which are not optimized

Other interesting questions :

- Why the belief in EQP did not pollute the synchrotron world ?
- Why refereed papers promoting the use of the EQP rule have been / are still published ?

Summary for discussion 1/2

1- The linac beams are out of the EQP theorem validity limit, to apply the "EQP rule" designing a linac is a mistake

2- The application of the "EQP rule" do not prevent emittance exchanges induced by coupling resonances

3- Safe tunes with beam footprints out of the coupling resonances can be found more easily when the "EQP rule" is not respected

Summary for discussion 2/2

4- The constraint imposed by the “EQP rule” on a linac design can lead to a non optimized beam dynamics and higher construction and operation costs

5- The question of energy exchange / emittance transfer must be analyzed as done in circular machines (tune diagram, evaluation of the excitation strength)

6- The “modern physics” tools developed to characterize the level of disorder (chaos) present in nonlinear Hamiltonian systems could be applied to characterize and optimize our beams (Service offer !)

讨论 时间

tǎolùn shíjiān

Discussion time …

The flea experiment

Our linac beams have noting to do with EQP then good and bad beam dynamics can be found whatever the EQP rule is or is not respected

To find an example of good beam dynamics when the EQP rule is respected and a bad one when it is not is not a demonstration of the EQP rule validity
(Ex : L.Y, APT design, LANL AOT-1:96-249, October 1996 and PAC97 proc.)

The flea experiment :

1- Put a flea on a table, ask it to jump, it jumps
2- Remove one leg to the flea, ask it to jump, it jumps
3- Remove the second leg, ask it to jump, the flea do not jump anymore
Conclusion : A flea becomes deaf when its two legs have been removed

Good experiment, wrong conclusion !

SNS working point shift to minimize the intra-beam striping effects
=> Safe working point without taking care of the EQP rule !