QUANTUM FEL I: MULTI-MODE THEORY

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Abstract

The quantum regime of the FEL in a single-mode, single-particle approximation is characterized by a two-level behaviour of the center-of-mass motion of the electrons. We extend this model to include all modes of the radiation field and analyze the effect of spontaneous emission. In particular, we investigate this scattering mechanism to derive experimental conditions for realizing an FEL in the quantum regime.

INTRODUCTION

In [1] the existence of the so-called quantum regime of the FEL was predicted and in [2] a quantum optics approach to the Quantum FEL (QFEL) was developed. Before we extend our model of the QFEL to a multi-mode theory, we briefly recapitulate its essential ingredients.

We start with the Hamiltonian of the FEL, i.e. an electron of mass m and charge e coupled to a single mode of the radiation field of frequency ω by the wiggler of amplitude $\mathcal{A}_W$. This Hamiltonian expressed in the Bambini-Renieri frame [3], which is a comoving non-relativistic frame of reference reads [4] in the interaction picture

$$\hat{H} \equiv \hbar g \hat{a}_L^\dagger \, \mathrm{e}^{-2ik\hat{z}} \, \mathrm{e}^{-\mathrm{i}\hat{\Delta}(\hat{p})t} + \mathrm{h.c.}\ . \tag{1}$$

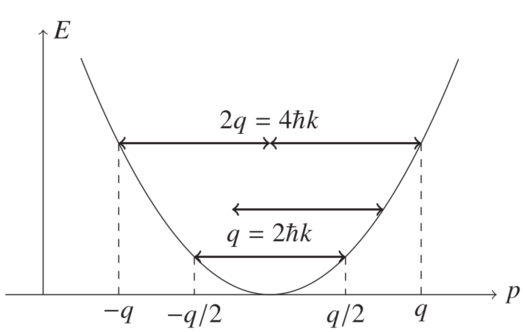

Figure 1: Momentum-energy parabola of the electron, that is $E = E(p)$. Only resonant photon transitions are of interest, the off-resonant ones can be neglected. Higher order photon transitions are on resonance as well, but they are suppressed by the quantum parameter α, defined by Eq. (5).

The coupling constant [2]

$$g \equiv \frac{e^2}{\hbar m}\sqrt{\frac{\hbar}{2\varepsilon_0 V\omega}}\mathcal{A}_W\ , \tag{2}$$

with the quantization volume V, Planck's constant $\hbar$, and the vacuum permittivity ε_0 couples the photon creation operator $\hat{a}_L^\dagger$ to the center-of-mass motion of the electron represented by the position and momentum operator, $\hat{z}$ and $\hat{p}$, respectively, where the detuning

$$\hat{\Delta}(\hat{p}) \equiv \hat{p}\,(2k/m) - \omega_r \tag{3}$$

contains the recoil frequency $\omega_r \equiv q^2/2m\hbar$. The operator $\exp[-2ik\hat{z}]$ shifts the electron momentum by the recoil

$$q \equiv 2\hbar k \tag{4}$$

determined by the wave number k.

At this point the Hamiltonian is exact for a strong classical wiggler field. In the classical regime the electron recoil is a negligible quantity, whereas in the quantum domain it enters in a crucial way into the detuning, which suppresses off-resonant photon transitions. This effect is illustrated in Fig. 1 where the initial and final momentum on the energy parabola are on resonance. For a single photon exchange, only the momentum states$|q/2\rangle$ and $|-q/2\rangle$ are on resonance.

Higher-order photon transitions, as seen in Fig. 1, are on resonance for higher momenta as well. However, it has been shown in [2] that a j-photon transition with $j > 1$ is proportional to α^j where

$$\alpha \equiv \frac{g\sqrt{n+1}}{\omega_r} \tag{5}$$

is the quantum parameter and n the photon number.

For $\alpha \ll 1$ only the single-photon transition remains, while higher-order transitions are suppressed. This feature represents the main criterion for the QFEL and the momentum states $|\pm q/2\rangle$ undergo a Rabi oscillation with the frequency

$$\Omega \equiv g\sqrt{n+1}\ . \tag{6}$$

This frequency determines the timescale on which a QFEL is operational, and will play a key role in our considerations.

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MULTI-MODE HAMILTONIAN

Since a QFEL behaves very much like a two-level system, a natural assumption is that spontaneous emission will also occur similar to that of a two-level atom [5], rather than to spontaneous undulator radiation in a classical FEL . Hence, we expect the probability $P(\boldsymbol{p}_0;t)$ to find the electron in its initial momentum state $|\boldsymbol{p}_0\rangle$ to decay exponentially as a function of time, that is

$$P(\boldsymbol{p}_0;t) = \mathrm{e}^{-\Gamma t}\, P(\boldsymbol{p}_0;0) \ . \tag{7}$$

Our goal is to find an expression for the decay constant Γ. In order to do so, we extend the Hamiltonian, Eq. (1), and couple a single electron to all modes. The resulting Hamiltonian then reads [6]

$$\hat{H}_{sp} \equiv \sum_{j,\lambda} \left(\boldsymbol{e}^*_{j,\lambda}\cdot\boldsymbol{e}_W\right)\left(\hbar g_j\right)\hat{a}^\dagger_j\, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\boldsymbol{q}_j\cdot\hat{\boldsymbol{r}}}\, \mathrm{e}^{-\mathrm{i}\hat{\Delta}_j(\hat{\boldsymbol{p}})t} + \text{h.c.} \ , \tag{8}$$

where $\boldsymbol{e}_{j,\lambda}$ and $\boldsymbol{e}_W$ denote the polarization vectors of the mode of the radiation field characterized by the mode indices j and λ, and of the wiggler field, respectively, and the coupling constant g_j between the electron and the jth mode.

The Hamiltonian $\hat{H}_{sp}$ differs from $\hat{H}$, Eq. (1), by the interaction of the electron with all modes, manifesting itself in a sum over all modes, and subsequently by the sum over all possible polarizations for each mode. The recoil

$$\boldsymbol{q}_j \equiv \hbar\left(\boldsymbol{k}_j - \boldsymbol{k}_W\right) \tag{9}$$

depends, in contrast to Eq. (4), on the wave vectors $\boldsymbol{k}_j$ of the jth mode and $\boldsymbol{k}_W$ of the wiggler. Moreover, the detuning

$$\hat{\Delta}_j(\hat{\boldsymbol{p}}) \equiv \frac{\hat{\boldsymbol{p}}\cdot\boldsymbol{q}_j}{\hbar m} - \frac{\boldsymbol{q}_j^2}{2m\hbar} - \left(\omega_j - \omega\right) \tag{10}$$

is now a slightly more complicated operator with the frequency ω_j of the jth mode.

Since in the one-mode Hamiltonian, Eq. (1), the angles in $\hat{\Delta}_j$ between $\boldsymbol{p}$ and $\boldsymbol{q}_j$ only take on the values 0 or π it corresponds to one space dimension Eq. (3). This restriction is no longer true for Eq. (8) and we have to take into account the directions of the involved wave vectors. The expression for the detuning is now dependent not only on the absolute value of the electron momentum, but also on its direction as well as the wave vectors $\boldsymbol{k}_j$ of all modes.

In order to find a resonance condition, we now simplify Eq. (10) significantly. The initial momentum is aligned with the z-axis, i.e. $\boldsymbol{p}_0 \equiv (q/2)\,\boldsymbol{e}_z = (\hbar\omega/c)\,\boldsymbol{e}_z$ to match the excited state of the QFEL, as discussed in the first section. When we apply the detuning operator $\hat{\Delta}(\hat{\boldsymbol{p}})$ defined by Eq. (10) onto the initial momentum state, we find,

$$\hat{\Delta}_j(\hat{\boldsymbol{p}})\,|\boldsymbol{p}_0\rangle = \left\{\frac{\hbar}{mc^2}\left[\omega\left(\omega_j\cos\delta - \omega\right) - \left(\omega_j-\omega\right)^2\right] - \left(\omega_j - \omega\right)\right\}|\boldsymbol{p}_0\rangle \ , \tag{11}$$

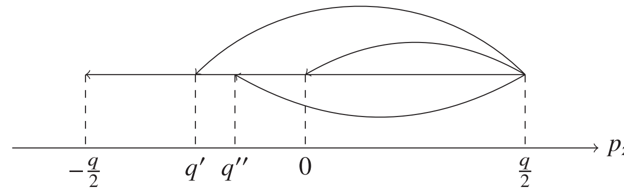

Figure 2: Additional resonances in the quantum regime of the FEL due to the presence of a reservoir. Whereas in Fig. 1 there is only a single resonance between $|\pm q/2\rangle$ remaining, with a reservoir any transition where the projection on the z-axis of the electron momentum $\boldsymbol{p}' \equiv \boldsymbol{p} - \boldsymbol{q}_j$ lies within $\pm q/2$ is possible.

where δ denotes the angle between $\boldsymbol{q}$ and $\boldsymbol{k}_j$.

The term associated with the square brackets scales with $\hbar\omega/mc^2 \ll 1$ and can therefore be neglected, which means that the frequency difference $\left(\omega_j - \omega\right)$ between mode and wiggler will determine the resonance condition. We recall that in Eq. (1) the electron momentum p determines the detuning and the frequency difference in the detuning vanishes due to only a single mode being present. This is no longer the case when many modes are relevant and it is indeed this frequency difference that dictates the resonance condition. We can therefore make the approximation

$$\mathrm{e}^{-\mathrm{i}\hat{\Delta}_j(\hat{\boldsymbol{p}})t}\,|\boldsymbol{p}\rangle \cong \mathrm{e}^{\mathrm{i}(\omega_j-\omega)t}\,|\boldsymbol{p}\rangle \ . \tag{12}$$

These additional resonances can be seen in Fig. 2. Once the electron has spontaneously emitted a photon and has therefore a new momentum $\boldsymbol{p}'$, then there is another mode on resonance due to Eq. (12), and the electron can spontaneously emit another photon, and so on. Indeed, the electron can spontaneously emit an infinite amount of photons since the wiggler, which we consider to be a classically device, is an external field.

When we now try to find the expression for the decay constant Γ the resulting differential equations will not decouple due to this cascade. We therefore have to find the point where this photon cascade stops.

PHOTON CASCADE

In order to cope with this problem we recall the relevant timescale of the QFEL as well as the condition for the strong coupling regime. For optimal gain the electron has to spend a time τ in the wiggler that allows for half of a Rabi cycle, that is

$$\Omega\tau = \frac{\pi}{2} \ . \tag{13}$$

In a conventional atom-reservoir interaction, we have for the strong-coupling regime the condition [7]

$$\frac{\Gamma}{g\sqrt{n+1}} \ll 1 \ , \tag{14}$$

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which yields with Eqs. (6) and (13) the inequality

$$\Gamma\tau \ll 1 \; . \tag{15}$$

Next we introduce the probability N_ℓ for an electron to spontaneously emit ℓ photons. The initial condition is that at $t = 0$ there are only electrons with momentum $\boldsymbol{q}/2$. When they are coupled to a reservoir, N_0 decays exponentially while $N_{\ell\neq 0}$ is increasing. For example, for a two-level atom, denoted with $\tilde{N}_\ell$, with the decay constant Γ_{at} we find [6]

$$\tilde{N}_0(t) = \mathrm{e}^{-\Gamma_{at}t}\, N_0(0) \approx (1-\Gamma_{at}t)\, N_0(0) \tag{16}$$

$$\tilde{N}_1(t) = \left(1-\mathrm{e}^{-\Gamma_{at}t}\right) N_0(0) \approx (\Gamma_{at}t)\, N_0(0) \tag{17}$$

where we have used the condition Eq. (14) for the strong coupling regime.

Since we are interested in more than a single spontaneous emission process, we now examine two subsequent spontaneous emission processes of the same electron. The respective set of differential equations read

$$\dot{N}_0(t) = -\Gamma N_0\,(t) \tag{18}$$

$$\dot{N}_1\,(t) = +\Gamma N_0\,(t) - \Gamma N_1\,(t) \tag{19}$$

$$\dot{N}_2\,(t) = +\Gamma N_1\,(t) \;\; . \tag{20}$$

Here we have assumed that the decay constant Γ is equal for all scattering events, which is reasonable since Γ, as we will show later, depends on the resonance condition which is independent on the electron momentum Eq. (12).

This set of equations is easily solved and with Eq. (15) we have $\exp[-\Gamma\tau] \approx 1 - \Gamma\tau$, and thus the solution reads

$$N_0(t) \approx (1-\Gamma\tau)\, N_0(0) \tag{21}$$

$$N_1(t) \approx [1-(\Gamma\tau)]\,(\Gamma\tau)\, N_0(0) \approx (\Gamma\tau)\, N_0(0) \tag{22}$$

$$N_2(t) \approx (\Gamma\tau)^2\, N_0(0) \approx 0 \; . \tag{23}$$

In comparison to Eqs. (16) and (17) we note only a small difference of the order of $\Gamma\tau$ which is not relevant on the timescale of our problem. This feature can be seen in Fig. 3 where the quantities N_0, N_1, N_2 are solved exactly and are plotted in comparison to the Rabi oscillations. The difference to $\tilde{N}_1$, Eq. (17), which represents only one spontaneously emitted photon does not matter until the first full Rabi cycle, but by that time the electron has already left the wiggler. We therefore conclude that we can neglect all further spontaneous emissions once the electron has emitted a photon. This behavior coincides perfectly with the two-level dynamics of the QFEL since in the quantum regime the electron can only emit a single photon.

DECAY CONSTANT: EXPLICIT FORM

With this information we now address the Schrödinger equation for the QFEL in the presence of a reservoir. The state vector $|\Psi(t)\rangle$ contains the electron momentum $\boldsymbol{p}$ as well as all possible photon numbers $\{n\}$ of all modes denoted by curly brackets, and reads

$$|\Psi(t)\rangle \equiv \int \mathrm{d}\boldsymbol{p}' \sum_n c\,(\boldsymbol{p}',\{n\}\,;t)\,|\boldsymbol{p}',\{n\}\rangle \tag{24}$$

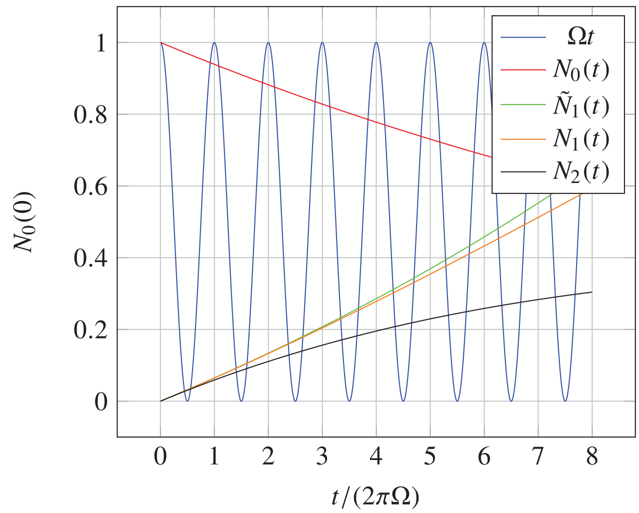

Figure 3: Timescale of the random scattering in the QFEL together with the corresponding Rabi oscillations for comparison. The value of the decay constant Γ was set to 1% of $g\sqrt{n+1}$, giving rise to the strong coupling regime. The quantity $\tilde{N}_1(t)$represents only a single scattering event and is the counterpart of the decay of a two-level atom.

Since we want to connect this state vector to Eq. (7) we calculate

$$P\,(\boldsymbol{p}_0;t) = |c\,(\hbar k_W \boldsymbol{e}_z,\{0\}\,;t)|^2 \;\; . \tag{25}$$

With the Hamiltonian, Eq. (8), the simplified detuning, Eq. (12) as well as the results Eqs. (17), (22) and (23) from the third section we arrive at the set of differential equations

$$\begin{aligned}\dot{c}\,(\hbar k_W \boldsymbol{e}_z,\{0\}\,;t) =& -\mathrm{i}\sum_{j,\lambda}\left(\boldsymbol{e}_W\cdot\boldsymbol{e}^*_{j,\lambda}\right) g_j\,\mathrm{e}^{\mathrm{i}(\omega_j-\omega)t}\\ &\times c\left(\hbar k_W\boldsymbol{e}_z - \boldsymbol{q}_j,\{1\}_j\,;t\right)\end{aligned} \tag{26}$$

$$\begin{aligned}\dot{c}\left(\hbar k_W \boldsymbol{e}_z - \boldsymbol{q}_j,\{1\}_j\,;t\right) =& -\mathrm{i}\left(\boldsymbol{e}^*_W\cdot\boldsymbol{e}_{j,\lambda}\right) g_j\,\mathrm{e}^{-\mathrm{i}(\omega_j-\omega)t}\\ &\times c\,(\hbar k_W\boldsymbol{e}_z,\{0\}\,;t)\end{aligned} \tag{27}$$

which can be solved by formally integrating Eq. (27) and substituting it into Eq. (26). The integral over time will result in a Dirac delta function that allows us to evaluate the sum. The resulting differential equation reads [7]

$$\dot{c}\,(\boldsymbol{p}_0,\{0\}\,;t) = 2\pi g^2\,(\omega)\,\mathcal{D}\,(\omega)\,c\,(\boldsymbol{p}_0,\{0\}\,;t) \tag{28}$$

with the mode density $\mathcal{D}\,(\omega) \equiv \omega^2/\left(\pi^2c^3\right)$ of the electromagnetic field [7].

The coupling constant g, defined inEq. (2), and the classical electron radius [8]

$$r_e \equiv \frac{e^2}{4\pi\varepsilon_0 mc^2} \tag{29}$$

together with the reduced Compton wavelength [8]

$$\lambdabar \equiv \frac{\hbar}{mc} \tag{30}$$

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allows us to express Eq. (28) as

$$\dot{c}\left(\boldsymbol{p}_0,\{0\};t\right) = -\frac{2}{3}\frac{r_e}{\lambda}a_0^2\omega \cdot c\left(\boldsymbol{p}_0,\{0\};t\right) \tag{31}$$

with the wiggler parameter [2]

$$a_0 \equiv \frac{\sqrt{2}e}{mc}\mathcal{A}_W\ . \tag{32}$$

Hence, we finally obtain the explicit expression

$$\Gamma \equiv \frac{1}{3}\frac{r_e}{\lambda}a_0^2\omega \tag{33}$$

for the decay constant appearing in Eq. (7).

DECAY CONSTANT: ESTIMATES

Expression Eq. (33) suggests that we can tune the decay rate Γ by the wiggler parameter a_0. In order to have a low decay the QFEL is best operational with

$$a_0 < 1\ . \tag{34}$$

Moreover, we have to ensure that we are in the strong coupling regime, that is the approximations from the third section hold true. Since $\Gamma = \Gamma(a_0,\omega)$ and $g = g(\alpha,\omega)$ we can directly calculate this ratio which reads

$$\frac{\Gamma}{g\sqrt{n+1}} = \frac{a_0^2}{\alpha}\frac{1}{\omega}\cdot 8.3\times 10^{15}\,\frac{1}{\mathrm{s}}\ . \tag{35}$$

However, we have to keep in mind that this ratio is still expressed in the Bambini-Renieri frame. We now connect this condition to the quantities in the laboratory frame and recall the transformation [3]

$$\omega = \frac{2\pi c}{\sqrt{\lambda_L\lambda_W}}\ . \tag{36}$$

of the resonance frequency to the laboratory system with the wiggler and laser wavelength, λ_W and λ_L, respectively.

The latter one is given by the identity [9]

$$\lambda_L = \frac{\lambda_W}{2\gamma^2}\left(1+a_0^2\right) \tag{37}$$

with the Lorentz factor

$$\gamma = \frac{E}{mc^2}\ , \tag{38}$$

which is determined by the electron energy E.

When we now combine Eqs. (36), (37) and (38) in Eq. (35) and evaluate all constants of nature we arrive at the condition

$$\chi \equiv a_0^2\sqrt{1+a_0^2}\,\frac{\lambda_W}{E} \ll \alpha 14.5\,\frac{\mathrm{mm}}{\mathrm{GeV}} \tag{39}$$

for the QFEL being in the strong coupling regime.

We compare the condition Eq. (39) in Table 1 for three FELs and see that none of them can fulfill it even for $\alpha \ll 1$. For an electron accelerator operating in the GeV regime, one does need at least a wiggler wavelength in the order of micrometers.

Table 1: The parameter χ defined by Eq. (39) providing the condition for a strong coupling regime of the QFEL for the FEL ELBE [10] at the HZDR as well as for the LCSL [11] at Stanford University. If χ exceeds the value of $\alpha 14.5$ mm/GeV with $\alpha \ll 1$, the coupling to the reservoir is too strong. The parameters a_0 and λ_W have been set to their possible minimum values and E to its possible maximum. Even if these devices are capable of reaching the quantum regime, none of them succeed at operating in the strong coupling regime.

FEL	χ in [mm/GeV]
ELBE U27	75.4
ELBE U100	82.2
LCLS	110.6

SUMMARY

In our previous model [2, 4] the ratio of the coupling constant g and the recoil frequency ω_r, i.e. the quantum parameter α, has served as the condition for the quantum regime of the FEL.

We are now in the position to incorporate decoherence due to a reservoir reflecting spontaneous emission as well. This extension has allowed us to derive a condition under which one does not only reach the quantum regime, but also the strong coupling regime of the QFEL which is best suited for a stable laser output.

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