THE PRESENT STATUS OF THE THEORY OF THE FEL-BASED HADRON BEAM COOLING*

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Abstract

The coherent electron cooling (CeC) [1] device is one of the new facilities under construction at Brookhaven National Laboratory (BNL). The CeC is a realization of the stochastic cooling with an electron beam serving as a pick-up and a kicker. Hadrons generate electron density perturbations in the modulator section, then these perturbations are amplified in an FEL, and then, they accelerate (or decelerate) hadrons in the kicker by their electric field with respect to the hadrons' velocities. Here we present the theoretical description of the modulator sector [2, 3], where the electron density perturbations are formed, and our new results on the time evolution of these perturbations in the FEL section, where they are amplified.

INTRODUCTION

The scheme of the CeC device is depicted in Fig. 1. It

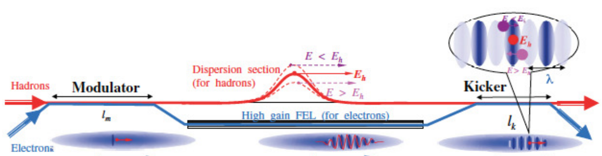

Figure 1: The scheme of the coherent electron cooler.

consists of three sections: the modulator section, where the electron density perturbations are created by hadrons, the FEL section, where these perturbations are amplified, and the kicker section, where the amplified perturbations accelerate (decelerate) hadrons moving slower (faster) than the one with the desired energy, before the kicker, hadrons pass through the dispersion section, where they are delayed in accordance with their energy deviations. In the present article, we describe theoretical models for all these sections.

THE MODULATOR SECTION

In the modulator section, each hadron in a hadron beam creates density perturbations in a co-propagating electron beam. The dynamical shielding of a charged particle in an infinite beam was considered in [4] and for the certain distribution the density perturbation was expressed as a one-dimensional integral, the more general method for a finite beam taking into account focusing fields was proposed in [2]. In the present article, we just quote some results from [3], where this problem was solved via the numerical evaluation of the inverse integral transforms for a variety of equilibrium distributions for a finite electron beam.

General Solution

Here we describe the dynamical shielding of a charged particle in an infinite isotropic electron beam via the Fourier and Laplace transforms. We introduce the following dimensionless variables:

$$\vec{\mathsf{x}} = \frac{\vec{x}}{r_{\mathrm{D}}}, \quad \vec{\mathsf{v}} = \frac{\vec{v}}{v_{\mathrm{rms}}}, \quad \mathsf{t} = \frac{t}{t_{\mathrm{p}}} \equiv t\omega_{\mathrm{p}}, \quad r_{\mathrm{D}} = \frac{v_{\mathrm{rms}}}{\omega_{\mathrm{p}}},$$

where

$$v_{\mathrm{rms}} = \sqrt{\frac{1}{\rho}\int v^2 f_0(\vec{v})d\vec{v}}, \quad \omega_{\mathrm{p}} = \sqrt{\frac{e^2\rho}{m_0\gamma\epsilon_0}}, \tag{1}$$

and the dimensionless equilibrium density $\mathsf{f}_0(\vec{\mathsf{v}})$:

$$f_0(\vec{v}) = \rho f_{\mathrm{d}}\mathsf{f}_0(\vec{\mathsf{v}}), \quad \int f_0(\vec{v})d\vec{v} = \rho. \tag{2}$$

For a unitary point charge moving along a straight line $\vec{\mathsf{y}}(\mathsf{t}) = \vec{\mathsf{x}}_0 + \vec{\mathsf{v}}_0\mathsf{t}$, we have for the induced electron density perturbation for any number of spacial dimensions d:

$$\mathsf{n}_1(\vec{\mathsf{x}},\mathsf{t}) = \mathsf{L}^{-1}\mathsf{F}^{-1}\left[\frac{\mathrm{e}^{-i\vec{\mathsf{k}}\cdot\vec{\mathsf{x}}_0}}{\left(\frac{f_{\mathrm{d}}^{-1}v_{\mathrm{rms}}^{-d}}{\mathsf{LF}_{\vec{\mathsf{k}}\mathsf{t}}(\mathsf{t}\mathsf{f}_0(\vec{\mathsf{v}}))}+1\right)\left(\mathsf{s}+i\vec{\mathsf{k}}\cdot\vec{\mathsf{v}}_0\right)}\right], \tag{3}$$

where $\mathsf{LF}_{\vec{\mathsf{k}}\mathsf{t}}(\mathsf{t}\mathsf{f}_0(\vec{\mathsf{v}}))$ depends on the equilibrium distribution:

$$\mathsf{LF}_{\vec{\mathsf{k}}\mathsf{t}}(\mathsf{t}\mathsf{f}_0(\vec{\mathsf{v}})) = \int_0^\infty \mathrm{e}^{-\mathsf{ts}}\mathsf{t}\int \mathsf{f}_0(\vec{\mathsf{v}})\,\mathrm{e}^{-i\vec{\mathsf{k}}\cdot\vec{\mathsf{v}}\mathsf{t}}d\vec{\mathsf{v}}d\mathsf{t}, \tag{4}$$

$f_{\mathrm{d}}^{-1}v_{\mathrm{rms}}^{-d}$ is a dimensionless factor, and L^{-1}, F^{-1} are the inverse Laplace and Fourier transforms, respectively.

Solutions for Some Distributions

We consider several distributions for the 1D, 2D and 3D cases. v_{rms}, f_{d}, and $f_{\mathrm{d}}^{-1}v_{\mathrm{rms}}^{-d}$ can be computed via (1) and (2) for all distributions excepting the Cauchy. The solution (3) is valid for all these cases, we only need to

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ISBN 978-3-95450-126-7

compute $\mathsf{LF}_{\vec{\mathsf{k}}\mathsf{t}}\left(\mathsf{tf}_0\left(\vec{\mathsf{v}}\right)\right)$ and $f_{\mathrm{d}}^{-1}v_{\mathrm{rms}}^{-d}$. For the Kapchinskij-Vladimirskij (KV) distribution we have:

$$\text{1D: } \mathsf{f}_0(\vec{\mathsf{v}}) = \delta(\mathsf{v}^2-1), \qquad \mathsf{LF}_{\vec{\mathsf{k}}\mathsf{t}}\left(\mathsf{tf}_0\left(\vec{\mathsf{v}}\right)\right) = \frac{\mathsf{s}^2-\mathsf{k}^2}{\left(\mathsf{s}^2+\mathsf{k}^2\right)^2},$$

$$\text{2D: } \mathsf{f}_0(\vec{\mathsf{v}}) = \frac{1}{\pi}\delta(\mathsf{v}^2-1), \qquad \mathsf{LF}_{\vec{\mathsf{k}}\mathsf{t}}\left(\mathsf{tf}_0\left(\vec{\mathsf{v}}\right)\right) = \frac{\mathsf{s}}{\left(\mathsf{s}^2+\mathsf{k}^2\right)^{\frac{3}{2}}},$$

$$\text{3D: } \mathsf{f}_0(\vec{\mathsf{v}}) = \frac{1}{2\pi}\delta(\mathsf{v}^2-1), \qquad \mathsf{LF}_{\vec{\mathsf{k}}\mathsf{t}}\left(\mathsf{tf}_0\left(\vec{\mathsf{v}}\right)\right) = \frac{1}{2}\hat{I}(\mathsf{s},\vec{\mathsf{k}},1),$$

where

$$\hat{I}(\mathsf{s},\vec{\mathsf{k}},\mathsf{v}) = \frac{i\mathsf{k}_3\mathsf{v}\left(\mathcal{S}_- - \mathcal{S}_+\right) + s\left(\mathcal{S}_- + \mathcal{S}_+\right)}{(\mathsf{s}^2+\mathsf{k}^2\mathsf{v}^2)\mathcal{S}_-\mathcal{S}_+},$$

$$\mathcal{S}_\pm = \sqrt{\left(\mathsf{s} \pm i\frac{\mathsf{k}_3^2\mathsf{v}}{\mathsf{k}}\right)^2},$$

for the water-bag (WB):

$$\mathsf{f}_0(\vec{\mathsf{v}}) = \frac{1}{2}\Theta(1-\mathsf{v}^2),\ \mathsf{LF}_{\vec{\mathsf{k}}\mathsf{t}}\left(\mathsf{tf}_0\left(\vec{\mathsf{v}}\right)\right) = \frac{1}{\mathsf{k}^2+\mathsf{s}^2},$$

$$\mathsf{f}_0(\vec{\mathsf{v}}) = \frac{1}{\pi}\Theta\left(1-\mathsf{v}^2\right),\ \mathsf{LF}_{\vec{\mathsf{k}}\mathsf{t}}\left(\mathsf{tf}_0\left(\vec{\mathsf{v}}\right)\right) = \frac{2}{\mathsf{k}^2}\frac{\sqrt{\mathsf{k}^2+\mathsf{s}^2}-s}{\sqrt{\mathsf{k}^2+\mathsf{s}^2}},$$

$$\mathsf{f}_0(\vec{\mathsf{v}}) = \frac{3}{4\pi}\Theta(1-\mathsf{v}^2),\ \mathsf{LF}_{\vec{\mathsf{k}}\mathsf{t}}\left(\mathsf{tf}_0\left(\vec{\mathsf{v}}\right)\right) = \frac{3}{2}\int_0^1 \mathsf{v}^2\hat{I}(\mathsf{s},\vec{\mathsf{k}},\mathsf{v})d\mathsf{v},$$

for 1D, 2D, and 3D respectively. For the Normal (Maxwell) distribution, $\mathsf{f}_0(\vec{\mathsf{v}}) = \pi^{-\frac{d}{2}}\mathrm{e}^{-\mathsf{v}^2}$, we have:

$$\mathsf{LF}_{\vec{\mathsf{k}}\mathsf{t}}\left(\mathsf{tf}_0\left(\vec{\mathsf{v}}\right)\right) = \frac{2}{\mathsf{k}^2}\left[1-\sqrt{\pi}\mathrm{e}^{\frac{\mathsf{s}^2}{\mathsf{k}^2}}\frac{s}{|\mathsf{k}|}\mathrm{Erfc}\frac{s}{|\mathsf{k}|}\right],$$

where $\mathrm{Erfc}(z)$ is the complementary error function,

$$v_{\mathrm{rms}} = \sqrt{\frac{dH_c}{2\beta}},\ f_{\mathrm{d}} = \left(\frac{\beta}{H_c}\right)^{\frac{d}{2}},\ f_{\mathrm{d}}^{-1}v_{\mathrm{rms}}^{-d} = (2/d)^{\frac{d}{2}}.$$

And for the Cauchy distribution we have:

$$\mathsf{f}_0(\vec{\mathsf{v}}) = \frac{\Gamma(\frac{1+d}{2})}{\Gamma(\frac{1}{2})\pi^{\frac{d}{2}}}\frac{1}{(1+\mathsf{v}^2)^{\frac{1+d}{2}}},\ \mathsf{LF}_{\vec{\mathsf{k}}\mathsf{t}}\left(\mathsf{tf}_0\left(\vec{\mathsf{v}}\right)\right) = \frac{1}{\left(\mathsf{s}+\mathsf{k}\right)^2},$$

$$v_{\mathrm{rms}} = \sqrt{\frac{H_c}{\beta}}, \quad f_{\mathrm{d}} = \left(\frac{\beta}{H_c}\right)^{\frac{d}{2}}, \quad f_{\mathrm{d}}^{-1}v_{\mathrm{rms}}^{-d} = 1.$$

Then the inverse integral transforms in (3) have to be computed. They can be rewritten as the discrete Fourier transforms and then evaluated numerically using the fast Fourier transform (FFT) algorithm. For the 1D Cauchy distribution, it is possible to compute these integral transforms analytically:

$$\begin{aligned}
\mathsf{n}_1\left(\vec{\mathsf{x}},\mathsf{t}\right) = {} & \frac{1}{4\pi}\frac{1}{\mathsf{v}_0-i}\left(\mathrm{e}^{-\mathcal{A}_+}\left(\mathrm{Ei}(\mathcal{A}_+)-\mathrm{Ei}(\mathcal{B}_+)\right)+\right.\\
& \left.+\mathrm{e}^{\mathcal{A}_+}\left(\mathrm{E}_1(\mathcal{A}_+)-\mathrm{E}_1(\mathcal{B}_+)\right)\right)+\\
& +\frac{1}{4\pi}\frac{1}{\mathsf{v}_0+i}\left(\mathrm{e}^{-\mathcal{A}_-}\left(\mathrm{Ei}(\mathcal{A}_-)-\mathrm{Ei}(\mathcal{B}_-)\right)+\right.\\
& \left.+\mathrm{e}^{\mathcal{A}_-}\left(\mathrm{E}_1(\mathcal{A}_-)-\mathrm{E}_1(\mathcal{B}_-)\right)\right), \quad (5)
\end{aligned}$$

$$\mathcal{A}_\pm = \frac{\mathsf{t}\mathsf{v}_0-\mathsf{x}+\mathsf{x}_0}{1\pm i\mathsf{v}_0}, \quad \mathcal{B}_\pm = \frac{\mathsf{x}_0-\mathsf{x}\pm i\mathsf{t}}{1\pm i\mathsf{v}_0},$$

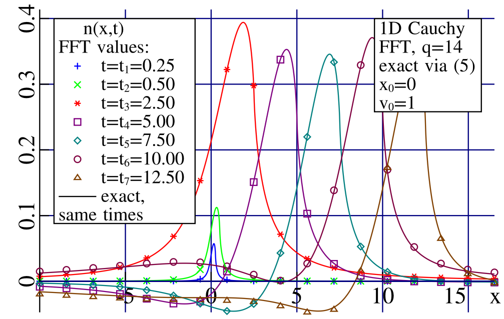

Figure 2: The exact and the FFT values for $n(\vec{\mathsf{x}},\mathsf{t})$ for the 1D Cauchy case.

and $\mathrm{E}_1(z)$ and $\mathrm{Ei}(z)$ are the exponential integral functions, which can be computed via the series expansions. In Fig. 2, it is shown that the numerical computations of the density perturbation using the FFT for the 1D Cauchy distribution are in a perfect agreement with the exact values computed via (5).

In Fig. 3, 4, we show the numerical results for $\mathsf{n}_1\left(\vec{\mathsf{x}},\mathsf{t}\right)$ obtained via the program we developed. q is a parameter defining the number of the grid points used in the FFT algorithm via $N = 2^q$. In Fig. 3, we show our numerical results for all 1D distributions we considered. The dynamics of the perturbations for the 2D KV, WB and Cauchy distributions is depicted on the Fig. 4, where we also show results for the 3D Cauchy distribution.

The method and the program we developed provide an opportunity to compute the dynamical shielding for many useful distributions for 1D, 2D and 3D infinite plasmas, they can be easily extended to any other equilibrium distributions including the empirical one. These results can be used for modeling the modulator section of CeC, they also can serve as a testing ground for other computational methods, e.g., particle-in-cell (PIC) simulations [5] and the more general numerical approach for a finite beam [3].

THE FEL SECTION

In the FEL section, the perturbations generated in the modulator are amplified via the high gain FEL. We apply the 1D FEL theory [6, 7] to derive an expression for the amplified perturbation density and the density corresponding to the self-amplified spontaneous emission (SASE), which can be used to estimate the saturation length, providing the limitations on the density perturbations amplification, and, as a result, on a performance of the whole CeC machine. We start with the coordinate transformation from the modulator to the FEL section, then describe the FEL system of

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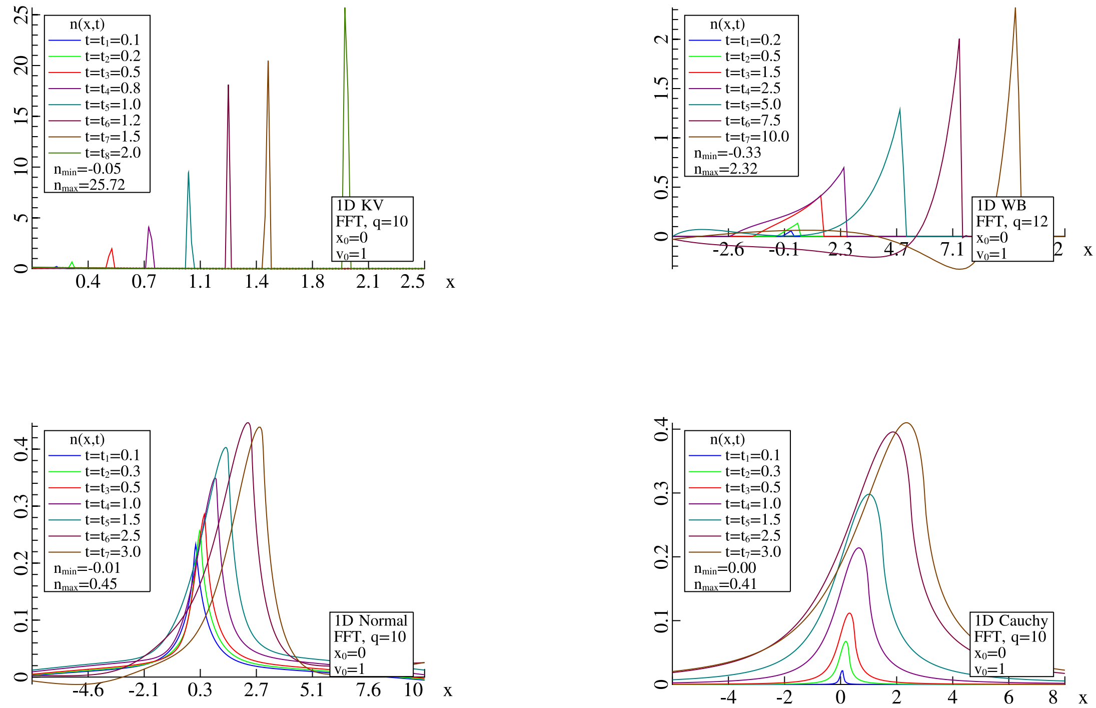

Figure 3: The density $n(\vec{x}, t)$ for the KV, WB, Normal, and Cauchy distributions in 1D.

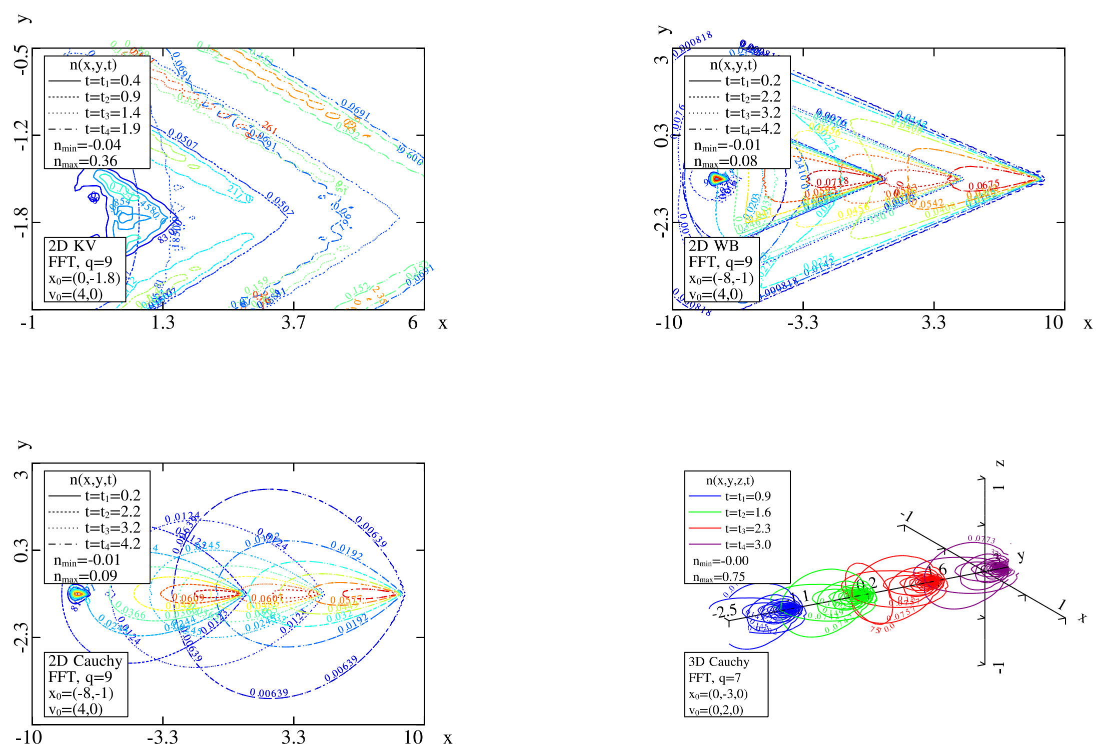

Figure 4: The density $n(\vec{x}, t)$ for the KV and WB in 2D and the Cauchy in 2D and 3D.

equations, and then solve it for the initial conditions corresponding to the perturbation from the modulator and the SASE.

From the Modulator to the FEL Amplifier

The modulator is described in a system of reference moving with the electron beam, however, the FEL theory is written in a laboratory frame, thus, we need to perform the Lorentz transformation, then we shift coordinates such that the ion have coordinates $(z, t) = (0, 0)$ by the end of the modulator section, then we introduce the standard independent variables in the FEL theory, (θ, z), a phase and a coordinate along the beam, via:

$$\begin{cases} \theta(z,t) = (k_1 + k_u)z + ck_1 t, \\ z(z,t) = z, \end{cases}$$

the phase θ is the position relative to the bunch center. The "reference electron" is the one that has $\theta = 0$, it has the same position as the ion. It is well-known that the phase-space density is Lorentz-invariant, thus, if we assume that the velocity distribution of the density perturbation is $\delta(v)$ and using the relation $\eta = \frac{v}{c}$ ($\eta = \frac{\gamma - \gamma_r}{\gamma_r}$ – the relative energy deviation from the resonance [7]), valid for the ultra-relativistic beams, we obtain the following relation between the density perturbation $\mathsf{n}_1^{(\text{lab})}(\theta, \mathsf{z})$ in the lab frame to be used as the initial perturbation in the FEL section and the density perturbation $\mathsf{n}_1(\vec{\mathsf{z}}', \mathsf{t}')$ in the beam's frame:

$$\mathsf{n}_1^{(\text{lab})}(\theta, \mathsf{z}) = \mathsf{n}_1\left(\mathsf{z}'(\theta,\mathsf{z}), \mathsf{t}'(\theta,\mathsf{z})\right),$$

and the discussed coordinate transformation is

$$\begin{cases} \mathsf{z}'(\theta,\mathsf{z}) & = \gamma\left(\mathsf{z} + \mathsf{L}_m - \beta\left(\frac{\theta - (\mathsf{k}_1 + \mathsf{k}_u)\mathsf{z}}{\mathsf{k}_1} + \mathsf{c}\mathsf{t}_i\right)\right), \\ \mathsf{t}'(\theta,\mathsf{z}) & = \frac{\gamma}{\mathsf{c}}\left(\frac{\theta - (\mathsf{k}_1 + \mathsf{k}_u)\mathsf{z}}{\mathsf{k}_1} + \mathsf{c}\mathsf{t}_i - \beta(\mathsf{z} + \mathsf{L}_m)\right), \end{cases}$$

where t_i is the time spent by the ion in the modulator, typically, it is of the order of ½ of the plasma osicllation:

$$\mathsf{t}_i \equiv \mathsf{t}_i(\vec{\mathsf{v}}_0') = \mathsf{L}_m \frac{\mathsf{c} + \beta\vec{\mathsf{v}}_0'}{\vec{\mathsf{v}}_0'\mathsf{c} + \beta\mathsf{c}^2},$$

L_m is the length of the modulator and $\vec{\mathsf{v}}_0'$ is the ion's velocity, other quantities used are standard in the FEL theory [7]. In all these formulas we used the dimensionless units introduced before and all vectors are one-dimensional.

The 1D Maxwell-Vlasov System

The slowly varying frequency domain amplitude of the radiation field $E_\nu(z)$ and the electron density distribution function $F(\theta, \eta, z)$, represented as a sum of a smooth background and a perturbation:

$$F(\theta, \eta, z) = F_0(\eta) + \delta F(\theta, \eta, z),$$

are governed by the 1D Maxwell-Vlasov equations:

$$\begin{cases} \left(\frac{\partial}{\partial z} + i\Delta\nu k_u\right) E_\nu(z) = -\chi_2 n_e \int d\eta \delta F_\nu(\eta, z), \\ \left(\frac{\partial}{\partial z} + 2k_u\eta\frac{\partial}{\partial\theta}\right)\delta F(\theta,\eta,z) = -\chi_1 \int E_\nu(z) \mathrm{e}^{i\theta\nu} d\nu \frac{d}{d\eta} F_0(\eta), \end{cases}$$

where we wrote $\delta F(\theta, \eta, z)$ in the Maxwell equation, as the smooth background doesn't contribute to the electric field. For other quantities used in the system we refer to [6, 7]. The continuity equation can be solved via the method of unperturbed orbits:

$$\delta F(\theta,\eta,z) = \delta F(\theta^{(0)}(0), \eta, 0) - \\ -\chi_1 \int_0^z \int E_\nu(z_1) \mathrm{e}^{i\theta^{(0)}(z_1)\nu} d\nu dz_1 \frac{d}{d\eta} F_0(\eta), \quad (6)$$

where the unperturbed orbit is given by:

$$\theta^{(0)}(z_1) = \theta + 2k_u\eta(z_1 - z).$$

Then we plug this expression into the Maxwell equation and solve it via the Laplace transform:

$$E_\nu(z) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{\mathrm{e}^{sz}}{D(s)} \Big[E_\nu(0) - \\ -\frac{\chi_2 n_e}{2\pi} \int_0^\infty \int \int \mathrm{e}^{-i\nu\theta - sz_2} \delta F(\theta^{(0)}(0), \eta, 0) d\theta d\eta dz_2\Big] ds, \quad (7)$$

where

$$D(s) = s + i\Delta\nu k_u - \bar{\rho}^3 \int \frac{i\nu}{(s + 2ik_u\eta\nu)^2} F_0(\eta) d\eta,$$

and $E_\nu(0)$ is the initial field, which we set to zero. To obtain the expression for the dynamics of the density perturbation we plug the expression (7) for $E_\nu(z)$ into the expression (6) for $\delta F(\theta, \eta, z)$. As the initial perturbation we can either consider the Klimontovich distribution function or the density perturbation formed in the modulator section. The first case corresponds to the SASE and the second one will give the dynamics of the amplification of the perturbation in the FEL section. We consider them in order.

The SASE

As the initial perturbation we consider the Klimontovich function:

$$\delta F(\theta^{(0)}(0), \eta, 0) = \frac{1}{N_\lambda} \sum_{j=1}^{N_e} \delta\big(\theta^{(0)}(0) - \theta_j\big)\delta\big(\eta - \eta_j\big),$$

where $N_\lambda = \lceil \lambda_1 \frac{I}{ec} \rceil$ is the number of electrons on one radiation wavelength and N_e is the number of electrons in a bunch. θ_j and η_j are the initial phases and energies of the electrons. For the θ-independent background distribution, θ_j is distributed uniformly over the bunch and η_j are the random variables with the distribution function $F_0(\eta)$. Following the route described in the previous subsection

ISBN 978-3-95450-126-7

we obtain for the SASE density perturbation:

$$\delta\mathsf{n}(\theta,\mathsf{z}) = \frac{1}{N_\lambda}\sum_{j=1}^{N_e}\delta\big(\theta - 2\mathsf{k}_u\eta_j\mathsf{z} - \theta_j\big) -$$

$$-\frac{\Gamma^3}{N_\lambda}\mathsf{F}^{-1}_{\nu,\theta}\mathsf{L}^{-1}_{\mathsf{s},\mathsf{z}}\sum_{j=1}^{N_e}\frac{i\mathrm{e}^{-i\nu\hat{\theta}_j(0)}}{2i\nu\mathsf{k}_u\hat{\eta}_j(0)+\mathsf{s}}\frac{\nu\mathrm{e}^{-\mathsf{sz}}}{\mathsf{D}(\mathsf{s})}\mathcal{I}_1(\mathsf{s},\mathsf{z},\nu,\mathsf{k}_u), \tag{8}$$

where

$$\mathcal{I}_1(\mathsf{s},\mathsf{z},\nu,\mathsf{k}_u) = \int\frac{1-\mathrm{e}^{\mathsf{z}(\mathsf{s}+2i\mathsf{k}_u\eta\nu)}+\mathsf{sz}+2i\mathsf{k}_u\mathsf{z}\eta\nu}{\mathrm{e}^{2i\mathsf{k}_u\mathsf{z}\eta\nu}(\mathsf{s}+2i\mathsf{k}_u\eta\nu)^2}F_0(\eta)d\eta,$$
$$\mathcal{I}_2(\mathsf{s},\nu,\mathsf{k}_u) = \int\frac{i\nu}{(\mathsf{s}+2i\mathsf{k}_u\eta\nu)^2}F_0(\eta)d\eta,$$
$$\mathsf{D}(\mathsf{s}) = \mathsf{s}+i\Delta\nu\mathsf{k}_u - \Gamma^3\mathcal{I}_2(\mathsf{s},\nu,\mathsf{k}_u), \tag{9}$$

and Γ is our equivalent of the Pierce parameter defined via $\Gamma = (2\chi_1\chi_2 r_{\mathrm{D}}^{-1}\mathsf{n}_e\mathsf{k}_u)^{\frac{1}{3}}$, this definition is different from the conventional one. In formula (8), we used the dimensionless units. The saturation is reached when the SASE perturbation is of the order of $\frac{N_e}{N_\lambda}$. This solution can be used for any equilibrium distribution $F_0(\eta)$. Here we apply it for the KV distribution $F_0(\eta) = \delta(\eta)$ and obtain:

$$\delta\mathsf{n}(\theta,\mathsf{z}) = \frac{1}{N_\lambda}\sum_{j=1}^{N_e}\delta\big(\theta - 2\mathsf{k}_u\eta_j\mathsf{z} - \theta_j\big) -$$

$$-\frac{\Gamma^3}{N_\lambda}\mathsf{F}^{-1}_{\nu,\theta}\mathsf{L}^{-1}_{\mathsf{s},\mathsf{z}}\sum_{j=1}^{N_e}\frac{\mathrm{e}^{-i\nu\hat{\theta}_j(0)}}{\mathsf{s}}\frac{i\nu\mathrm{e}^{-\mathsf{sz}}}{\mathsf{D}(\mathsf{s})}\frac{(1-\mathrm{e}^{\mathsf{zs}}+\mathsf{sz})}{\mathsf{s}^2},$$

$$\mathsf{D}(\mathsf{s}) = \mathsf{s}+i\Delta\nu\mathsf{k}_u - \Gamma^3\frac{i\nu}{\mathsf{s}^2},$$

the inverse Laplace transform can be easily computed as the sum over the residues at the roots of the denominator and the inverse Fourier transform can be computed numerically via the FFT.

The Smooth Density Perturbation

As the initial perturbation we can also use the perturbation formed in the modulator:

$$\delta n(\theta,0) = \mathsf{n}_1^{(\mathrm{lab})}(\theta,\mathsf{z}=0),$$

and

$$\delta F(\theta^{(0)}(0),\eta,0) = \delta n\big(\theta - 2k_u\eta z,0\big)F_0(\eta).$$

Following the procedure described in the beginning of this section, after quite lengthy computations we obtain:

$$\delta\mathsf{n}(\theta,\mathsf{z}) = \mathsf{n}_1\left(\mathsf{z}'(\theta-2\mathsf{k}_u\eta\mathsf{z},0),\mathsf{t}'(\theta-2\mathsf{k}_u\eta\mathsf{z},0)\right) +$$
$$+\frac{\tilde{\chi}_{12}\mathsf{n}_e}{i}\mathsf{L}^{-1}_{\mathsf{s},\mathsf{z}}\mathsf{F}^{-1}_{\nu,\theta}\mathsf{F}^{-1}_{\vec{\mathsf{k}},\frac{L_m}{r_{\mathrm{D}}\gamma}}\left\{\int\frac{\mathsf{M}(\vec{\mathsf{k}},\mathsf{s}_1^*(\vec{\mathsf{k}}))\mathrm{e}^{i\nu\mathsf{k}_1(\mathsf{ct}_i-\beta\mathsf{L}_m)}}{\frac{2\mathsf{k}_u\eta\gamma}{\mathsf{k}_1}\left(\frac{\mathsf{s}_1^*(\vec{\mathsf{k}})}{\mathsf{c}}+\vec{\mathsf{k}}\beta\right)+\mathsf{s}}\times\right.$$
$$\left.\times\frac{F_0(\eta)}{\mathsf{D}(\mathsf{s})}d\eta\int\frac{1-\mathrm{e}^{2i\eta_1\mathsf{k}_u\nu\mathsf{z}-\mathsf{sz}}}{2i\eta_1\mathsf{k}_u\nu+\mathsf{s}}\frac{d}{d\eta_1}F_0(\eta_1)d\eta_1\right\}, \tag{10}$$

where $\tilde{\chi}_{12}$ and n_e are the dimensionless equivalents of $\chi_1\chi_2$ and n_e, the electron volume density, respectively, $\mathsf{M}(\vec{\mathsf{k}},\mathsf{s})$ is just the expression in square brackets in (3), $\mathsf{s}_1^*(\vec{\mathsf{k}})$ is defined via:

$$\mathsf{s}_1^*(\vec{\mathsf{k}}) = \frac{it_{\mathrm{p}}ck_1}{\gamma}\left(\frac{\vec{\mathsf{k}}\gamma\beta}{r_{\mathrm{D}}k_1}+\nu\right) = i\nu\frac{t_{\mathrm{p}}ck_1}{\gamma}+i\vec{\mathsf{k}}\frac{t_{\mathrm{p}}c\beta}{r_{\mathrm{D}}},$$

$\mathsf{D}(\mathsf{s})$ is the same as in (9). $\mathsf{F}^{-1}_{\vec{\mathsf{k}},\frac{L_m}{r_{\mathrm{D}}\gamma}}$ denotes the inverse Fourier transform over $\vec{\mathsf{k}}$ evaluated at $\frac{L_m}{r_{\mathrm{D}}\gamma}$. The expression (10) is valid for any equilibrium distribution $F_0(\eta)$. For the KV distribution $F_0(\eta) = \delta(\eta)$, we have:

$$\delta\mathsf{n}(\theta,\mathsf{z}) = \mathsf{n}_1\left(\mathsf{z}'(\theta,0),\mathsf{t}'(\theta,0)\right) -$$
$$-\Gamma^3\mathsf{F}^{-1}_{\nu,\theta}\mathsf{L}^{-1}_{\mathsf{s},\mathsf{z}}\mathsf{F}^{-1}_{\vec{\mathsf{k}},\frac{L_m}{\gamma}}\left[\frac{(1+\mathsf{sz})\mathrm{e}^{-\mathsf{sz}}-1}{\mathsf{s}^3\mathsf{D}(\mathsf{s})}\times\right.$$
$$\left.\times\,\mathsf{M}\left(\vec{\mathsf{k}},i\nu\frac{\mathsf{ck}_1}{\gamma}+i\vec{\mathsf{k}}\mathsf{c}\beta\right)\nu\mathrm{e}^{i\nu\mathsf{k}_1(\mathsf{ct}_i-\beta\mathsf{L}_m)}\right],$$

as for the SASE case, the inverse Laplace transform can be easily computed as the sum over the residues at the roots of the denominator and the inverse Fourier transforms can be computed numerically via the FFT.

THE KICKER

In the kicker, the hadrons interact with the electric field produced by their own amplified density perturbations [8]. Mathematically, the problem is very similar to the one solved for the modulator, but, in the kicker, as initial density perturbation we have the hadron's charge and the amplified electron density from the FEL section. Solving for the Fourier image of the density perturbation, we can easily obtain the potential of the field in the Fourier domain. And then, doing the inverse Fourier transform we can compute the potential in the space domain.

RESULTS AND DISCUSSION

In the present article, we proposed a theoretical model for the coherent electron cooling device. The modulator section is described in the infinite beam approximation and solved via the inverse integral transforms [3]. For the 1D Cauchy distribution, it is possible to compute these transforms analytically, for other distributions they can be computed via the FFT, there is a perfect agreement of the analytical and numerical results for a variety of parameter's values. The FEL section is described in the framework of the 1D FEL theory, it is possible to extend this description to the 3D FEL theory. The expressions for the amplified density perturbation and the SASE contribution are derived. Both of them are written using the inverse integral transforms, which can be computed numerically in the same way as it was done for the modulator section. The computation of the SASE contribution allows one to estimate the limitations of the amplification of the density modulation in the FEL section. The kicker can be described in a very similar way as the modulator. This completes our model of the CeC. The numerical methods required for the computations for the every section are thoroughly tested (see Fig. 2 and [3]) in the modulator section and, undoubtedly, can be applied in other sections.

ACKNOWLEDGMENTS

Various discussions with Gang Wang are gratefully acknowledged.

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ISBN 978-3-95450-126-7