A GENERAL METHOD FOR ANALYZING 3-D EFFECTS IN FEL AMPLIFIERS

Panagiotis Baxevanis, Ronald D.Ruth, Zhirong Huang
SLAC National Accelerator Laboratory, Menlo Park, CA 94025, USA

Abstract

FEL configurations in which the parameters of the electron beam vary along the undulator become relevant when considering new aspects of existing FELs or when exploring novel concepts. This paper describes a fully three-dimensional, analytical method suitable for studying such systems. As an example, we consider a seeded FEL driven by a beam with varying transverse sizes. In the context of the Vlasov-Maxwell formalism, a self-consistent equation governing the evolution of the radiation field amplitude is derived. An approximate solution to this equation is then obtained by employing an orthogonal expansion technique. This approach yields accurate estimates for both the amplified power and the radiation beam size. Specific numerical results are presented for two different sets of X-ray FEL parameters.

INTRODUCTION

The standard approach for analyzing the physics of a high-gain FEL in the linear regime is based on formulating and solving the eigenmode problem for the system [1–4]. However, this method is only valid when the electron beam parameters do not depend on the longitudinal position z. In some important cases, this assumption is not satisfied [5]. In this paper, we develop an analytical technique that is applicable even when such a z-dependence is present and use it to study an FEL that is driven by a mismatched or an unfocused beam. For full details, we refer to [6].

THEORY

To begin with, let us assume that the FEL radiation is generated as a relativistic, bunched electron beam passes through a planar, parabolic-pole-face undulator with symmetric focusing. From Ref. [7], the 3-D single particle equations of motion are

$$\frac{d\mathbf{x}}{dz} = \mathbf{p}\,, \quad \frac{d\mathbf{p}}{dz} = -k_\beta^2\mathbf{x}\,, \tag{1}$$

$$\frac{d\theta}{dz} = 2k_u\eta - \frac{k_r}{2}[\mathbf{p}^2 + k_\beta^2\mathbf{x}^2]\,, \tag{2}$$

$$\frac{d\eta}{dz} = \kappa_1\int_0^\infty d\nu E_\nu(\mathbf{x},z)e^{-i\Delta\nu k_u z}e^{i\nu\theta} + c.c.\,, \tag{3}$$

where $\mathbf{x}$ is the transverse position, k_β is the total focusing strength of the undulator system, $\theta = (k_u + k_r)z - \omega_r t + [k_r K^2/(8k_u\gamma_0^2)]\sin(2k_u z)$ is the electron phase, $\lambda_u = 2\pi/k_u$ is the undulator period, $\lambda_r = \lambda_u(1 + K^2/2)/(2\gamma_0^2) = 2\pi/k_r = 2\pi c/\omega_r$ is the resonant wavelength, γ_0 is the average Lorentz factor, K is the undulator parameter, $\eta = (\gamma - \gamma_0)/\gamma_0$ is the relative energy deviation, $\nu = \omega/\omega_r$ is the scaled frequency, $\Delta\nu = \nu - 1$, E_ν is the complex amplitude of the radiation field, $\kappa_1 = eKJJ/(4\gamma_0^2 m_0 c^2)$ (e and m_0 are the electron charge and mass, $JJ = J_0[K^2/(4+2K^2)] - J_1[K^2/(4+2K^2)]$) and c.c. stands for complex conjugate. Up to the onset of saturation effects, the operation of the FEL is accurately described by the following set of coupled, frequency-domain, linearized Vlasov-Maxwell equations:

$$\frac{\partial f_\nu}{\partial z} + \mathbf{p}\frac{\partial f_\nu}{\partial \mathbf{x}} - k_\beta^2\mathbf{x}\frac{\partial f_\nu}{\partial \mathbf{p}} + i\theta' f_\nu = -\kappa_1 E_\nu e^{-i\Delta\nu k_u z}\frac{\partial f_0}{\partial \eta}\,, \tag{4}$$

$$\left(\frac{\partial}{\partial z} + \frac{\nabla_\perp^2}{2ik_r}\right)E_\nu(\mathbf{x},z) = -\kappa_2 e^{i\Delta\nu k_u z} \times \int d^2\mathbf{p}\int d\eta f_\nu(\eta,\mathbf{x},\mathbf{p},z)\,, \tag{5}$$

where $f_\nu = \int d\theta f_1 e^{-i\nu\theta}/(2\pi)$ is the amplitude of the perturbation f_1 to the distribution function of the electron beam, f_0 is the background distribution, $\kappa_2 = eKJJ/(2\varepsilon_0\gamma_0)$ and $\theta' = d\theta/dz$ is given by Eq. (2). Moreover, the unperturbed distribution f_0 - which we take to be θ-independent - evolves according to the zeroth-order Vlasov equation

$$\frac{\partial f_0}{\partial z} + \mathbf{p}\frac{\partial f_0}{\partial \mathbf{x}} - k_\beta^2\mathbf{x}\frac{\partial f_0}{\partial \mathbf{p}} = 0 \tag{6}$$

and its normalization is given by $\int d^2\mathbf{p}\int d^2\mathbf{x}\int d\eta f_0 = N_b/l_b$, where l_b and N_b are, respectively, the length of one bunch and the number of electrons it contains. We note that this analysis dose not include space charge or shot noise effects.

Equation for the amplitude of the radiation field

The general solution of Eq. (4) is

$$f_\nu = f_\nu(z=0)\exp\left(-i\frac{d\theta}{dz}z\right) - \kappa_1\frac{\partial f_0}{\partial \eta}\int_0^z d\zeta E_\nu(\bar{\mathbf{x}},\zeta)e^{-i\Delta\nu k_u\zeta}\exp\left(i\frac{d\theta}{dz}\xi\right)d\zeta\,, \tag{7}$$

where $\xi = \zeta - z$ and $\bar{\mathbf{x}} = \mathbf{x}\cos(k_\beta\xi) + (\mathbf{p}/k_\beta)\sin(k_\beta\xi)$. One can also show that Eq. (6) admits solutions of the form $f_0 = f_0(\eta, \mathbf{x}\cos\bar{z} - (\mathbf{p}/k_\beta)\sin\bar{z}, \mathbf{x}k_\beta\sin\bar{z} + \mathbf{p}\cos\bar{z})$, where $\bar{z} = k_\beta z_0$, $z_0 = z - z_e$ and z_e is a constant. We choose a background distribution given by

$$f_0 = \frac{N_b}{(2\pi)^{5/2}l_b\sigma^2\sigma'^2\sigma_\eta}\exp\left(-\frac{\eta^2}{2\sigma_\eta^2}\right)\times \exp\left(\frac{k_\beta\Gamma\sin(2k_\beta z_0)}{2\sigma'^2}\mathbf{x}\mathbf{p} - \frac{k_\beta^2[1+\Gamma\cos^2(k_\beta z_0)]}{2\sigma'^2}\mathbf{x}^2 - \frac{1+\Gamma\sin^2(k_\beta z_0)}{2\sigma'^2}\mathbf{p}^2\right)\,, \tag{8}$$

ISBN 978-3-95450-123-6

where $\Gamma = \sigma'^2/(\sigma^2 k_\beta^2) - 1$ is the mismatch parameter. The above expression corresponds to a mismatched beam with a round, Gaussian profile. The oscillating electron beam size is $\sigma_e(z) = \sigma[1 + \Gamma \sin^2(k_\beta z_0)]^{1/2}$. In this paper, we will only consider the case in which the electron beam is initially unmodulated, so $f_\nu(z = 0) = 0$. Substituting Eq. (7) into Eq. (5), performing the integration over η and changing the momentum integration variable from $\mathbf{p}$ to $\bar{\mathbf{x}}$, we obtain an integro-differential equation for the radiation field amplitude E_ν:

$$\frac{\partial E_\nu}{\partial z} + \frac{\nabla_\perp^2 E_\nu}{2ik_r} = \int d^2\bar{\mathbf{x}} \int_0^z d\zeta \Lambda(\mathbf{x}, \bar{\mathbf{x}}, z, \zeta) E_\nu(\bar{\mathbf{x}}, \zeta)\,, \tag{9}$$

where

$$\Lambda(\mathbf{x}, \bar{\mathbf{x}}, z, \zeta) = -\frac{4i\rho^3 k_u^3}{\pi\sigma'^2} \xi e^{-i\Delta\nu k_u \xi - 2\sigma_\eta^2 k_u^2 \xi^2} \times \tag{10}$$
$$\exp\left(-\frac{k_\beta^2[1 + ik_r\sigma'^2\xi + \Gamma\sin^2(k_\beta\zeta_0)]}{2\sigma'^2 \sin^2(k_\beta\xi)} \mathbf{x}^2 \right.$$
$$- \frac{k_\beta^2[1 + ik_r\sigma'^2\xi + \Gamma\sin^2(k_\beta z_0)]}{2\sigma'^2\sin^2(k_\beta\xi)} \bar{\mathbf{x}}^2 + \frac{k_\beta^2}{\sigma'^2} \times$$
$$\left. \frac{(1 + ik_r\sigma'^2\xi)\cos(k_\beta\xi) + \Gamma\sin(k_\beta z_0)\sin(k_\beta\zeta_0)}{\sin^2(k_\beta\xi)} \mathbf{x}\bar{\mathbf{x}} \right)$$

with $\zeta_0 = \zeta - z_e$ - we recall that $z_0 = z - z_e$ and $\xi = \zeta - z$ - and

$$\rho = \left(\frac{K^2 JJ^2}{16\gamma_0^3 k_u^2 \sigma^2} \frac{I}{I_A} \right)^{1/3} \tag{11}$$

is the Pierce parameter [7], expressed in terms of the peak current $I = eN_b c/l_b$ and the Alfven current $I_A = 4\pi\varepsilon_0 m_0 c^3/e \approx 17\,\text{kA}$.

Expansion method

Our goal is to obtain a solution to Eq. (9) that is consistent with a specified initial amplitude $E_\nu(\mathbf{x}, z = 0)$. The method we adopt is based on expanding the field amplitude in terms of a complete set of orthogonal basis functions. The basis we employ consists of *generalized* Gauss-Laguerre transverse modes

$$\psi_{nm}(\mathbf{x}, z) = \left(\frac{n!}{(n + |m|)!} \right)^{1/2} \left(\frac{\sqrt{2}r}{w} \right)^{|m|}$$
$$\times L_n^{|m|}\left(\frac{2r^2}{w^2} \right) \psi_{00}(\mathbf{x}, z) e^{im\phi} e^{-i(2n+|m|)u}\,, \tag{12}$$

where (r, ϕ) are polar coordinates in the transverse plane, (n, m) are integers with $n \geq 0$ and $L_n^{|m|}$ are the associated Laguerre polynomials. Here,

$$\psi_{00}(\mathbf{x}, z) = \left(\frac{k_r\beta_1}{\pi} \right)^{1/2} \frac{1}{z - i\beta} \exp\left(\frac{ik_r r^2}{2(z - i\beta)} \right) \tag{13}$$

is the fundamental basis mode - defined through a complex-valued function $\beta = \beta_1 + i\beta_2 = \beta(z)$ - while $w = (2/(k_r\beta_1))^{1/2} |z - i\beta|$ and $u = \tan^{-1}((z + \beta_2)/\beta_1)$ are, respectively, the spot size and Gouy phase associated with it. The basis elements described above satisfy the orthonormality condition $\langle \psi_{nm} \mid \psi_{pq} \rangle \equiv \int \psi_{nm}^* \psi_{pq} d^2\mathbf{x} = \delta_{np}\delta_{mq}$ and reduce to the standard vacuum modes of paraxial optics when β is a constant. For simplicity, we assume that the external seed consists of a *finite* number of vacuum Gauss-Laguerre modes with the same azimuthal index m and - constant - basis parameter β_s. In view of the axial symmetry of the problem, we seek a solution with an $e^{im\phi}$ angular dependence. The expansion for the field amplitude is then

$$E_\nu(\mathbf{x}, z) = \epsilon_\nu \sum_{n=0}^{\infty} C_{nm}(z) \psi_{nm}(\mathbf{x}, z)\,, \tag{14}$$

where C_{nm} are chosen to be dimensionless and ϵ_ν is a constant. Inserting Eq. (14) into Eq. (9) yields an infinite set of coupled evolution equations for the expansion coefficients:

$$\frac{dC_{nm}}{dz} = (2n + |m| + 1) \frac{iC_{nm}}{2\beta_1} \frac{d\beta_2}{dz}$$
$$+ \sqrt{n(n + |m|)} \frac{C_{n-1,m}}{2\beta_1} \frac{d\beta}{dz}$$
$$- \sqrt{(n + 1)(n + |m| + 1)} \frac{C_{n+1,m}}{2\beta_1} \frac{d\beta^*}{dz}$$
$$+ \int_0^z d\zeta \sum_{p=0}^{\infty} C_{pm}(\zeta) \Lambda_{pm}^{nm}(z, \zeta, \beta, \beta_\zeta) \tag{15}$$

where

$$\Lambda_{pm}^{nm}(z, \zeta, \beta, \beta_\zeta) = (-1)^{p+n+1}$$
$$\times \frac{8i\rho^3 k_u^3}{D} \frac{(p + n + |m|)!}{(n!p!)^{1/2}[(p + |m|)!(n + |m|)!]^{1/2}}$$
$$\times \left(\frac{\beta_{1\zeta}}{\beta_1} \right)^{\frac{|m|+1}{2}} \frac{(z - i\beta)^{n+|m|}}{(\zeta - i\beta_\zeta)^{p+|m|}} \frac{(\zeta + i\beta_\zeta^*)^p}{(z + i\beta^*)^n}$$
$$\times \xi e^{-i\Delta\nu k_u \xi - 2\sigma_\eta^2 k_u^2 \xi^2} \frac{(X - Y)^p (X - 1)^n}{X^{p+n+|m|}} \frac{d^p b^{|m|}}{a^{p+|m|}}$$
$$\times {}_2F_1(-p, -n; -p - n - |m|; J)\,. \tag{16}$$

In the relations given above, ${}_2F_1$ is a Gaussian hypergeometric function, $\beta_\zeta \equiv \beta(\zeta)$ and $\beta_{1\zeta} \equiv \beta_1(\zeta) = \mathrm{Re}[\beta_\zeta]$. Moreover,

$$a = 1 + \Gamma\sin^2(k_\beta z_0) + ik_r\sigma'^2 \left(\xi - \frac{\sin^2(k_\beta\xi)}{k_\beta^2(\zeta - i\beta_\zeta)} \right)\,, \tag{17}$$

$$d = a - \frac{2k_r\sigma'^2\beta_{1\zeta}\sin^2(k_\beta\xi)}{k_\beta^2|\zeta - i\beta_\zeta|^2}\,, \tag{18}$$

$$b = (1 + ik_r\sigma'^2\xi)\cos(k_\beta\xi) + \Gamma\sin(k_\beta z_0)\sin(k_\beta\zeta_0)\,, \tag{19}$$

$$Y = \frac{\beta_{1\zeta}}{\beta_1} \frac{|z - i\beta|^2}{|\zeta - i\beta_\zeta|^2} \frac{b^2}{ad}\,. \tag{20}$$

ISBN 978-3-95450-123-6

Defining

$$D_1 = -ik_r\sigma'^2 \sin^2(k_\beta\xi)/k_\beta^2 + [1+\Gamma\sin^2(k_\beta z_0) + ik_r\sigma'^2\xi](\zeta - i\beta_\zeta)\,, \tag{21}$$

$$D_2 = k_r\sigma'^2\xi - i[1+\Gamma\sin^2(k_\beta\zeta_0)] + k_\beta^2\left(\frac{1}{k_r\sigma'^2} + i\xi\right)(1+\Gamma+ik_r\sigma'^2\xi)(\zeta - i\beta_\zeta)\,, \tag{22}$$

we also have $D = (iD_1 + (z+i\beta^*)D_2)/(2\beta_1)$, $D_3 = D_1/(z-i\beta)$, $X = D/D_3$ and $J = 1 - Y/((X-Y)(X-1))$. For a monochromatic input signal, the amplified power and radiation beam size are given by

$$P(z) = P_0 \sum_{n=0}^{\infty} |C_{nm}(z)|^2 = \int I(\mathbf{x}, z) d^2\mathbf{x} \tag{23}$$

and

$$\sigma_r^2(z) = \frac{\int r^2 I(\mathbf{x},z)d^2\mathbf{x}}{2\int I(\mathbf{x},z)d^2\mathbf{x}} = \left(\frac{w^2}{4}\right)\times \frac{1}{\sum\limits_{n=0}^{\infty}|C_{nm}(z)|^2}\left\{\sum_{n=0}^{\infty}(2n+|m|+1)|C_{nm}(z)|^2 - 2\,\mathrm{Re}[e^{2iu}\sum_{n=1}^{\infty}\sqrt{n(n+|m|)}C_{n-1,m}(z)C_{nm}^*(z)]\right\}, \tag{24}$$

where P_0 is the input power and $I \propto |E_\nu|^2$ is the intensity of the radiation. We note that we have set $\epsilon_\nu = (\int d^2\mathbf{x}|E_\nu(\mathbf{x},0)|^2)^{1/2}$ so that $\sum_{n=0}^{\infty}|C_{nm}(0)|^2 = 1$. From the above, it is evident that numerically solving an appropriately truncated version of the set of Eq. (15) can lead to valuable quantitative information about the FEL radiation. To proceed, we need to specify the basis function $\beta(z)$. By choosing $\beta(z=0) = \beta_s$, we ensure that $C_{nm}(0) = 0$ for all $n > M$, where M is the maximum radial index of the seed modes. Thus, it is reasonable to approximate the field amplitude using the first $M+1$ modes in the expansion of Eq. (14) ($0 \le n \le M$). To improve the accuracy of our calculation, we follow a self-consistent approach in which the basis function evolves in correlation with the expansion coefficients [8]. If we select β so that

$$\sqrt{(M+1)(M+|m|+1)}\,\frac{C_{Mm}}{2\beta_1}\frac{d\beta}{dz} + \int_0^z d\zeta \sum_{p=0}^{M} C_{pm}(\zeta)\Lambda_{pm}^{M+1,m}(z,\zeta,\beta,\beta_\zeta) = 0\,, \tag{25}$$

it can be shown that the coefficient of the next order mode ($\psi_{M+1,m}$) vanishes identically. Eq. (25), along with a truncation of Eq. (15), define our approximation scheme.

NUMERICAL RESULTS

To illustrate our method, we have used two different FEL parameter sets, both of which correspond to hard X-ray machines (Table 1). Set 1 roughly describes the current operating parameters of the LCLS while Set 2 refers to a machine with lower beam energy and more ambitious undulator specifications. For Set 1, we consider two configurations: one with a matched electron beam ($\Gamma = 0$) and one for which the beam is underfocused, with $z_e = 0$, $\sigma/\sigma_m = \sqrt{2.5}$ and $\Gamma = \sigma_m^4/\sigma^4 - 1 = -0.84$. In both cases, we assume a Gaussian seed with the following parameters: input beta $\beta(z=0)/\beta_m = 0.38 + i0.21$, detune $\hat{\nu}_m = \Delta\nu/(2\rho_m) = -0.38$ and input power $P_0 = 2\,\mathrm{kW}$. In Figs. 1-3, we show the comparison between the results obtained with our technique and GENESIS simulation data. In the linear regime, the theoretical results are in good agreement with simulation, even though only a single Gaussian mode has been used in obtaining the former. Including higher order modes in our calculation makes the comparison with simulation even more favorable. In the case of the matched beam, our analytical solution almost exactly reproduces the variational value for the fundamental FEL growth rate in the exponential-gain region. For Set 2, we explore two distinct cases: we either assume a conventional undulator that relies solely on natural focusing or consider a device with the same K and λ_u but with no focusing. The second option can refer to an RF undulator, where the transverse defocusing effect is typically very weak. The case with no focusing can be treated by the formalism we have developed by taking the limit $k_\beta \longrightarrow 0$. The resulting analytical expressions refer to an FEL that is driven by a coasting beam with a single waist at $z = z_e$. We choose the waist beta function $\beta_e \equiv \sigma/\sigma'$ to be equal to the natural value of 13.78 m, which is fairly close to the optimum beta for these parameters. We then use our method to study the influence of the waist position upon the total FEL gain for an undulator length $L_u = 2.5\beta_e = 34.45\,\mathrm{m}$ and compare the results with the total gain for the case of a beam that is matched to the conventional undulator. For all these runs, we have assumed a Gaussian seed with $\beta(z=0)/\beta_m = 0.92 + i0.91$ and $\hat{\nu}_m = -0.94$. The data obtained - again using a single mode approximation - are shown in Fig. 4. From the latter, we conclude that one can recover as much as 93% of the gain for the matched beam when $z_e = 1.25\beta_e = 0.5L_u$, i.e. by placing the

Table 1: Undulator and electron beam parameters

Parameter	**Set 1**	**Set 2**
Undulator parameter K	3.7	0.5
Undulator period λ_u	3 cm	0.5 cm
beam energy $\gamma_0 mc^2$	14.31 GeV	2.21 GeV
Resonant wavelength λ_r	1.5 A°	1.5 A°
Peak current I	3 kA	3 kA
Energy spread σ_η	10^{-4}	10^{-4}
Normalized emittance $\gamma_0\epsilon$	0.5 μm	0.5 μm
Matched beta $\beta_m = 1/k_\beta$	30 m	13.78 m
Matched beam size σ_m	23.14 μm	39.89 μm
ρ_m (for $\sigma = \sigma_m$)	5.4×10^{-4}	2.3×10^{-4}
External focusing	Yes	No

ISBN 978-3-95450-123-6

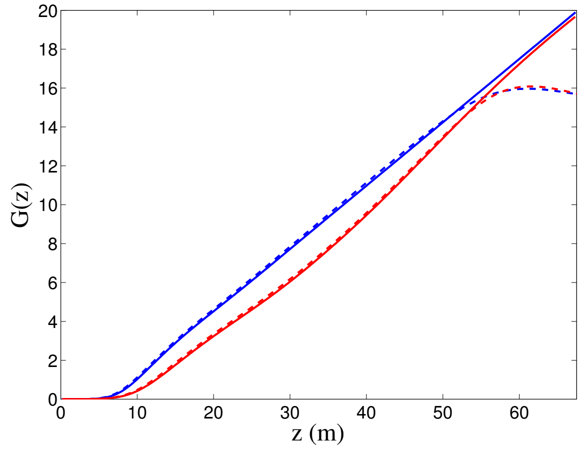

Figure 1: FEL gain $G(z) = \log[P(z)/P_0]$ for a matched (blue curves) and a mismatched beam (red curves) - LCLS parameters. Both analytical results (solid curves) and simulation data (dashed curves) are shown.

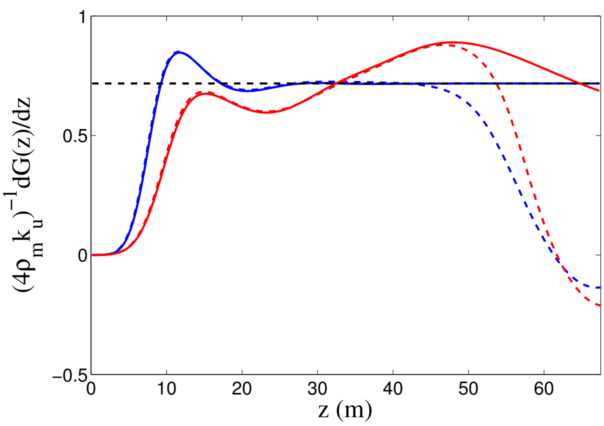

Figure 2: Normalized growth rate for a matched and a mismatched beam (same legend as in Fig. 1). The black line refers to the growth rate of the fundamental FEL mode.

waist of the unfocused beam in the middle of the undulator segment.

CONCLUSION

An expansion method has been developed for solving the initial value problem of an FEL with variable electron beam parameters, taking full account of 3-D effects. We have used this technique in the study of various aspects of the operation of a high gain FEL that is driven by a beam with varying transverse sizes. The results obtained are in good agreement both with simulation and with the theoretical prediction for the special case of a matched electron beam. The ability to describe the radiation field with a small number of expansion modes makes this method potentially useful for parameter studies in the linear regime.

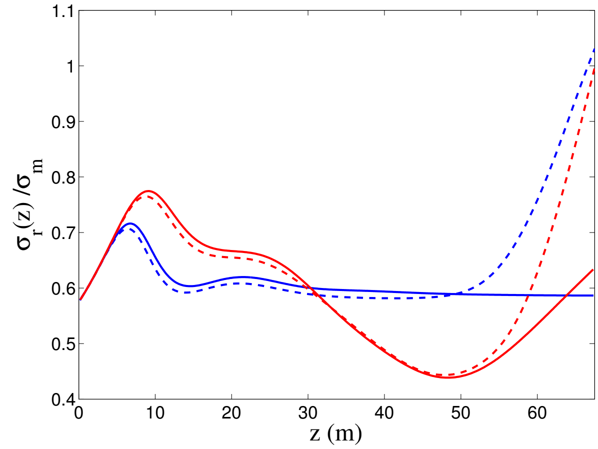

Figure 3: Scaled radiation beam size for a matched and a mismatched electron beam (same legend as in Fig. 1).

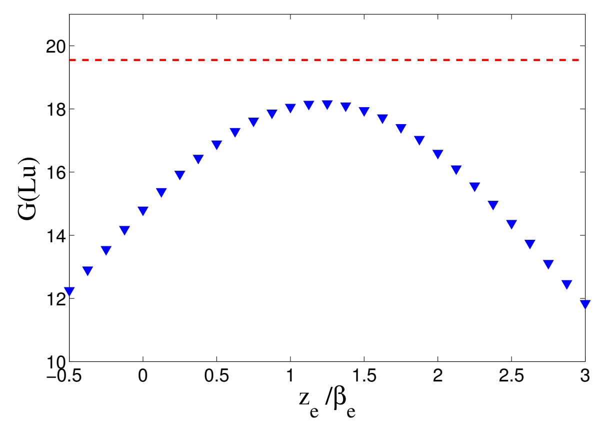

Figure 4: Total gain vs scaled waist position (blue markers) for an FEL driven by an unfocused beam (Set 2 parameters). Also shown is the gain for a matched beam (red dashed line).

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ISBN 978-3-95450-123-6