Simple Physics for Marvelous Light:
FEL Theory Tutorial

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undulators and free electron lasers

undulator

fel oscillator

high-gain, single-pass fel

magnetic undulator

(\(N\) periods)

relativistic electron beam

\(E = \gamma mc^2\)

\(\lambda_u\)

\(\lambda_x\)

\(2\theta\)

diffraction limited cone of x-rays

\(\lambda_x \approx \frac{\lambda_u}{2\gamma^2}\)

\(\theta \approx \frac{1}{\gamma \sqrt{N}}\)

permanent magnets

(smco)

flux concentrating steel (fe) pole pieces

fel theory tutorial aug 2011 kjk
Spontaneous emission by one e⁻ in undulator

- The e⁻ emits EM wave in the forward direction due to its x-acceleration. Consider the wave fronts from successive undulator periods:

  - The e⁻ is slower since (1) $c > v \approx c(1-1/2\gamma^2)$, and (2) its trajectory is curved.
  - The distance the EM wave slips ahead of the e⁻ in one undulator period is the wavelength of the spontaneous emission:
    - $\lambda_1 = \lambda_u (1+K^2/2)/2\gamma^2$
  - The length of the spontaneous emission for an $N_u$ period undulator is
    - $\Delta z_{rad} = N_u \lambda_1$
A coherent wavetrain of length $\Delta z = N_1 \lambda_1$

- At $z = 0$: $E(t) = E_0 \exp(i \omega_1 t), \quad \omega_1 = \frac{2\pi c}{\lambda_1}, \quad -\frac{\Delta z}{2c} < t < \frac{\Delta z}{2c}$

- Frequency domain: $\tilde{E}(\omega) = \int dt \exp(-i\omega t)E_0(t)$

$$\tilde{E}(\omega) = \text{const} \times N_1 \frac{\sin \pi N_1 \Delta \nu}{\pi N_1 \Delta \nu}, \quad \Delta \nu = \frac{\omega - \omega_1}{\omega_1}$$

- Spectral intensity $dW(\omega)/d\omega \sim |\tilde{E}(\omega)|^2 \sim (\sin x/x)^2$ peaked around with relative bandwidth $\Delta \nu = \Delta \omega/\omega \sim 1/N_1$

- For undulator radiation, the electron energy may be off by $\Delta \gamma$. Then $x = \pi N_u (\Delta \nu - 2\Delta \gamma/\gamma)$
Undulator radiation from a collection of electrons—a “bunch”

- The wave trains from $N_e$ electrons in a bunch of length $\Delta z_{el}$ combine to “chaotic light” of length $\Delta z$ consisting of coherent “spikes” of length $\Delta z_{rad}$

\[
\Delta z = \sqrt{\Delta z_{el}^2 + \Delta z_{rad}^2} = \sqrt{\Delta z_{el}^2 + (\lambda_1 N_u)^2}
\]

- Phase space area:

\[
\Delta \Omega_t = \Delta z \times \Delta \omega / \omega = \sqrt{\lambda^2 + (\Delta z_{el} / N_u)^2} \geq \lambda_1
\]

- Temporal coherence: if $\Delta \Omega_t \sim \lambda_1$, then the front and back of the radiation pulse can be brought together for interference.
A monochromator increases temporal coherence

A monochromator extends a wavetrain: \( \Delta \omega \rightarrow \Delta \omega_M \ll \Delta \omega, \frac{\omega_1}{\Delta \omega_M} = N_M \)

- A collection of wavetrains becomes coherent \( \Delta \Omega \rightarrow \lambda_1 \) if \( \Delta z_{el} / N_M \ll \lambda_1 \)

\[
\Delta \Omega_{t,M} = \sqrt{\lambda_1^2 + \left( \frac{\Delta z_{el}}{N_M} \right)^2}
\]

- However, the intensity of wavetrains in general add incoherently
- The amplitudes add in phase if \( \Delta z_{el} \ll \lambda_1 \) or if electrons are concentrated at positions \( z = n\lambda_1, \ n=1,2,.. \)

- This is what FELs are about!
**Energy exchange between e\(^-\) and coherent radiation**

- An e\(^-\) and a coherent EM wave travel through the undulator. They can exchange energy due to the electron’s x-velocity. However, the net exchange over many periods vanishes because of the velocity mismatch.

- However, when the EM wavelength is \(\lambda_1\) electrons see the same EM field in the successive period and the energy exchange can accumulate.

- An e\(^-\) arriving at \(A_0\) loses energy to the field (e\(v \mathbf{E} \ < 0\)). Similarly for e\(^-\) s nearby and at distances \(n\lambda_1\), \(n=1,2,...\) also lose energy. However, those at \(\lambda_1/2\) away gain energy.

- The electron beam develops energy modulation (period length \(\lambda_1\)).

- Higher energy electrons are faster \(\rightarrow\) density modulation develops.

- Coherent EM of wavelength \(\lambda_1\) is generated \(\rightarrow\) FEL gain.
Equations for electron motion

- **Variables**
  - $z =$ distance along the undulator axis
  - $\theta =$ electron position in the moving bunch in units of $2\pi/\lambda$
  - $\eta =$ electron relative energy: $(\gamma - \gamma_0)/\gamma_0$

- **EM wave:** $E_x = \hat{E} \cos(kz - \omega t)$

- **Pendulum equations**
  
  $\frac{d\theta_i}{dz} = \left(\frac{4\pi}{\lambda_u}\right)\eta_i$ : higher energy $e$ moves faster
  
  $\frac{d\eta_i}{dz} = \frac{eK}{2\gamma^2mc^2} \hat{E} \sin \theta_i$ : $\theta$-dependent energy gain

- The electrons’ energy loss averaged over initial $\theta$ becomes the gain in the EM field
Spontaneous & stimulated emission, and gain

- Total energy conservation: \( \mathcal{E}_i + \int |E_L|^2 = \mathcal{E}'_i + \int |E'_L + e_{s,i}|^2 \)
- Laser energy conservation: \( \int |E_L|^2 = \int |E'_L|^2 \)
- \( \mathcal{E}'_i = \mathcal{E}_i - 2 \text{Re} \int e_{s,i} * E'_L - \int |e_{s,i}|^2 : 2 \text{Re} \int e_{s,i} * E'_L = W_{stim} \)
- The amplitude \( e_{s,i} \) depends on electron energy. Since, \( \mathcal{E}_i \neq \mathcal{E}'_i \), we guess that it depends on the average: \( \bar{\mathcal{E}}_i = \mathcal{E}_i - 0.5 \times W_{stim} \)
- Thus \( e_{s,i}(\mathcal{E}_i) \rightarrow e_{s,i}(\mathcal{E}_i - 0.5 W_{stim}) = e_{s,i}(\mathcal{E}_i) - (\int e_{s,i} * E'_L)(\partial e_{s,i}/\partial \mathcal{E}_i) \)
- The last term adds to \( E'_L : E'_L \rightarrow (1 + g)E'_L \), \( g = -\frac{1}{2} \frac{\partial}{\partial \mathcal{E}} \sum_i |e_{s,i}|^2 \)
- **Gain is the derivative of the spontaneous emission spectrum**
**Madey’s theorem for power gain**

\[
G = -\pi \frac{I}{emc^2} \frac{\partial}{\partial \gamma} \left( \frac{dW_{spon}(\omega, \gamma)}{d\omega} \right)
\]

- \(dW_{spon}(\omega, \varepsilon)/d\omega = \) the spectral density of spontaneous emission, \(I = \) the electron current

The FEL gain BW is about half of the spontaneous emission BW
Synchronism: The spacing between e bunches = 2L/n (L = cavity length)

Lasing starts if: \((1 + G_0) R_1 R_2 > 1\) (\(R_{1,2}\) : mirror reflectivity), \(G_0\) = initial gain

- Need high-reflectivity, normal-incidence mirrors
- Could be difficult for soft x-ray and shorter wavelengths?

The gain \(G\) decreases as the intracavity power increases. A steady state ("saturation") is reached when \((1 + G_{sat}) R_1 R_2 = 1\)

Output power = \((1 - R_{1,2}) \cdot\) intracavity power – loss in mirrors

FELOs have been built for IR, visible, and UV wavelengths
Existing and future FEL oscillators

Jlab IR-UV User Facility

Possible future hard x-ray FEL oscillator using diamond crystals for x-ray cavity

E = 150 MeV
135 pC pulses up to 75 MHz
(20)/120/1 microJ/pulse in (UV)/IR/THz
250 nm – 14 microns, 0.1 – 5 THz

Sources are simultaneously produced for pump-probe studies

Built primarily under ONR and AF funding to investigate high power FEL operation

UV system is presently under construction

By varying the incidence angle $\Theta$, one can obtain a wide range of photon energies that satisfy Bragg’s law $E = E_H \cos \Theta$

Tunability allows one to pick a single crystal for all wavelengths of interest
Low gain FEL oscillator performance: Intensity

- FEL efficiency:
  - Spontaneous radiation spectral width $\frac{\Delta \omega}{\omega} = 2\frac{\Delta \gamma}{\gamma} \sim \frac{1}{N_u}$
  - We have seen: $\frac{\Delta \gamma}{\gamma}|_{FEL} = \frac{1}{2} \frac{\Delta \gamma}{\gamma}|_{spon} \approx \frac{1}{4N_u}$

- **FEL output power** $\approx \left(\frac{1}{4N_u}\right) \times$ electron beam power
- Electron energy spread requirement: $\frac{\Delta \gamma}{\gamma}|_{spread} \leq \frac{1}{4N_u}$
**Temporal and spectral evolution**

- As the roundtrip pass number $n$ increases
  - The spectral width decreases: $\Delta \omega / \omega \propto 1 / \sqrt{n}$
  - The pulse width decreases: $\Delta z \propto 1 / \sqrt{n}$
- Evolution stops when $\Delta z \times \Delta \omega / \omega \rightarrow \lambda$
- $\rightarrow$ The limiting spectral width (the super-mode theory)

$$\Delta \omega / \omega \rightarrow \sqrt{\frac{1}{2N_u \Delta z}} \frac{\lambda}{\Delta z_0} = \text{(gain BW} \times \text{“transform limited BW”)}^{1/2}$$

- However, the full transform limit $\Delta \omega / \omega = \lambda / \Delta z_0$ may be achieved with nonlinear saturation:

**XFEL0**: $\lambda / \Delta z \sim 10^{-7}$ for $\lambda = 1$ Å and $\Delta z = 1$ ps
High-gain, single-pass FELs

- With high electron beam qualities and a **long undulator**, the single pass gain can be made very high
- If coherent seed is available, then amplifier mode with harmonic generation produces intense, coherent, short wavelength output
- Without a seed, SASE (self-amplified-spontaneous–emission) produce quasi-coherent output

SACLA undulator, 110 m
Exponential growth in high-gain FEL

- Three quantities characterizing the amplification process
  - \( E \): EM amplitude:
  - \( b = \langle \exp(-i \theta) \rangle \): Density modulation:
  - \( P = \langle \eta \exp(-i \theta) \rangle \): Energy modulation:

- Evolution as a function of \( z \)
  - EM field induces energy modulation: \( \frac{dP}{dz} = 2k_u C_1 E \)
  - Energy modulation induces density modulation: \( \frac{db}{dz} = -i 2k_u P \)
  - Density modulation generates EM field: \( \frac{dE}{dz} = 2k_u C_2 b \)

\[ \frac{d^3E}{dz^3} = -i (2k_u)^3 C_1 C_2 E ; \quad \rho^3 = C_1 C_2 = \frac{1}{8\pi} \frac{I}{I_A} \frac{K^2}{1+K^2} \frac{\nu \lambda^2}{\Sigma_A} \]

- The solution is \( E = \sum a_i \exp(-2i \mu_i \rho k_u z) \):
  - \( \mu_i \) are solutions of \( \mu^3 = 1 \)
  - \( \mu_1 = 1, \mu_2 = \frac{-1 - \sqrt{3}i}{2} \), \( \mu_3 = \frac{-1 + \sqrt{3}i}{2} \)
  - The root \( \mu_3 \) gives rise to an exponential growth
General 1-D solution in linear regime

- Slowly varying amplitude in frequency domain $E_\nu(z)$, $\nu = \Delta \omega / \omega_1$

- The solution of the initial value problem for an arbitrary momentum distribution $V(\eta)$ is:
  
  $$E_\nu(z) = \sum_j \frac{e^{-2i\mu_j \rho k_u z}}{D'(\mu_j)} \left[ E_\nu(0) + i \frac{eK^* n}{8\varepsilon_0 \gamma \rho k_u N} \sum_{j=1}^{N_e} \frac{e^{-i\nu \theta_j(0)}}{\eta_j(0) / \rho - \mu_j} \right], \quad D'(\mu) = \frac{dD(\mu)}{d\mu}$$

- The first term is coherent amplification and the second SASE
- Here $\mu_j$ is the solution of ($\Delta \nu = \text{"detuning"}= \nu - 1 \ll 1$):
  
  $$D(\mu) = \mu - \frac{\Delta \nu}{2\rho} - \int d\eta \frac{V(\eta)}{(\eta / \rho - \mu)^2} = 0$$

- For cold beam at $\Delta \nu = 0$: $\mu - 1/\mu^2 = 0$ (reproduce the cubic equation)
The main characteristics of high-gain FEL in terms of $\rho$

- For a typical x-ray FEL, $\rho \sim 10^{-3}$
- Power gain length: $L_G \sim \lambda_u/(4\pi\rho)$
- Undulator periods for saturation: $\sim 1/\rho$
- Spectral bandwidth: $\Delta\omega/\omega \sim \rho$
- Coherence length: $l_c \sim \lambda/\rho \ll \Delta z_e$  
  $\rightarrow$ “Chaotic” light
- Saturation power: $\sim \rho \times$ beam power
- Maximum amplification factor: $\sim \#$ of electrons in $l_c$
Transverse properties of propagating EM waves

- Coherent EM wave diffracts if transversely restricted by $\Delta x$:
  \[ \Delta \phi = \frac{\lambda}{\Delta x} \rightarrow \Delta \Omega_x = \Delta x \Delta \phi = \lambda \] (1)

- An extended source of length $L$ has a depth-of-field blur:
  \[ \Delta x = \Delta \phi L \] (2)
  \[ \Delta x = \sqrt{\lambda L} \] and $\Delta \phi = \sqrt{\lambda / L}$

- For fundamental Gaussian mode:
  \[ \sigma_r = \sqrt{\frac{\lambda}{4\pi} Z_R} \quad \sigma_{r'} = \sqrt{\frac{\lambda}{4\pi} \frac{1}{Z_R}} \quad \sigma_r \sigma_{r'} = \frac{\lambda}{4\pi} \]

- Note the correspondence between the light and e-beam optics:
  \[ \frac{\lambda}{4\pi} \leftrightarrow \varepsilon_x \text{ (rms emittance)} \]
  \[ Z_R \text{ (Rayleigh length)} \leftrightarrow \beta-\text{function} \]
An incoherent light can be made coherent by selecting a phase space area of $\lambda$. Transversely coherent light exhibits interference.
Transverse properties of undulator radiation

- Red shift at an angle $\phi$, $\lambda_1(\phi) = \lambda_u (1 + \phi^2 \gamma^2 + K^2/2)/2\gamma^2$

- Angular divergence of central cone $\sigma_{\Delta \phi} = \sqrt{\lambda_1/2L_u} = \sqrt{\left(\frac{\lambda_1}{4\pi}\right)\left(\frac{2\pi}{L_u}\right)}$

- The undulator radiation by one-electron is approximately the transversely coherent Gaussian mode with $Z_R \sim L_u/2\pi$

- Undulator radiation from a beam of electrons is a convolution of the one-electron radiation and the electron beam distribution

→ Transversely coherent if $\varepsilon_x \leq \lambda$
Temporal bunching implies transverse coherence $\rightarrow$ FELs are transversely coherent

- In low gain oscillator, the mode is similar to free-propagating Gaussian laser mode
- In high-gain FEL the mode is guided
Low gain FEL oscillator is transversely coherent

- The eigenmode is an approximately free-propagating Gaussian mode with $Z_R \sim L_u/2\pi$ and the waist in the middle of the undulator
- Temporally coherent along the full length of the pulse $\rightarrow$ transversely coherent
- FELO is coherent transversely as well as temporally
Transverse behavior in high-gain FELs

- The 1D dispersion relation becomes the eigenmode equation for the complex growth rate $\mu$ and the associated transverse mode shape. For parallel electron beam with the density profile $U(x)$:

$$\left(\mu - \frac{\Delta \nu}{2 \rho} + \Lambda^2 \nabla_T^2 - U(x) \int d\eta \frac{V(\eta)}{(\mu - \eta / \rho)^2}\right) A(x) = 0$$

- The electron’s betatron motion can be included.
- The initial value problem can be solved as a sum of eigenmodes.
- **A single mode dominates in the exponential growth regime** → Frozen (or guided) transverse profile of mode size $\sim \sqrt{(\lambda/4\pi) L_G}$
X-ray diffraction pattern of a single Mimivirus particle imaged at the LCLS. (Image courtesy of Tomas Ekeberg, Uppsala University.)
The progression of marvelous x-ray sources from spectral brightness point of view
Additional topics (in x-ray FEL)

- **Harmonic generation for coherent soft x-ray**
  - Echo-enabled scheme for high harmonics

- **Improving the temporal coherence of high-gain x-ray FEL**
  - Use input coherence radiation via harmonic generation (FEL, laser, or both)
  - Filter SASE by a monochromator and send through another high gain FEL—self seeding scheme

- **Ultra short pulse (~100 atto-second) generation**
  - single-cycle energy modulation → compression → subfemto-second current spike → enhanced FEL growth rate

- **Other bright ideas YOU might invent, triggered by this tutorial!**
Thank you for the Program Committee for the invitation, and Those who helped me to prepare this talk

- Ryan Lindberg
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- .....
Thank you!