COLLECTIVE AND INDIVIDUAL ASPECTS OF FLUCTUATIONS IN RELATIVISTIC ELECTRON BEAMS FOR FREE ELECTRON LASERS*

K.-J. Kim† and R. R. Lindberg‡, ANL, Argonne, IL 60439, USA

Abstract

Fluctuations in highly bright, relativistic electron beams for free electron lasers (FELs) may exhibit both collective as well as individual particle aspects, similar to that of non-relativistic plasmas. If the collective part characterized by the plasma oscillation dominates, then it is feasible to suppress shot noise (density fluctuations) as is well known in microwave devices. However, individual particle aspects become more significant as we consider fluctuations of shorter wavelengths. To study this issue, we solve the 1-D coupled Gauss-Klimontovich equations by the Laplace transform technique. We find the density fluctuations to be composed of a linear combination of the collective plasma oscillations and the random motion of Debye-screened dressed particles. The relative magnitude of the random to the collective part scales with $k\lambda_D$, where k is the fluctuation wavenumber and λ_D is the Debye length suitably defined for relativistic beams. Electron beams used to generate x-ray self-amplified spontaneous emission (SASE) typically have $k\lambda_D \sim 1$, and therefore collective oscillations are not relevant. When $k\lambda_D$ is small the decrease in shot noise (density fluctuation) after one quarter plasma period is accompanied by an increase in "momentum noise" which scales as $1/k\lambda_D$. Since the effective seed for SASE includes both terms, a reduction in shot noise may not result in a reduction of the SASE power.

INTRODUCTION

Fluid models for electron beams are typically sufficient to compute mean field averages and the properties of long-scale fluctuations. However, since fluid models smooth out the intrinsic graininess of individual electrons, they ignore fluctuations associated with discrete particle effects. Thus, it is in general important to understand to what degree discrete particle effects modify those predictions based on continuum models. Since electron density fluctuations are the source of self-amplified spontaneous emission (SASE) in free-electron lasers (FELs), it is particularly important to understand both aspects of particle motion if one wishes to compute how these density fluctuations evolve and eventually seed an FEL. Such fluctuation characteristics have been studied extensively for non-relativistic plasmas [1, 2, 3]. Here, we study the role of collective and individual particle effects in relativistic beams with particular focus on their implications to FEL performance.

We begin our discussion by introducing the governing equations and method of solution, which uses the method of Ref. [4] first introduced to analyze linear evolution in high-gain free electron lasers. We then show how the electron beam density (or bunching factor) can be decomposed into two components, one that contains the collective particle motion due to the electron plasma wave, and the other that contains terms associated with single particle dynamics. The latter can be understood as the free motion of Debye-screened electrons. Finally, we compute the relative contribution of these two terms for LCLS parameters, and show to what extent collective plasma wave dynamics are important for FEL-type electron beams.

PARTICLE EQUATIONS AND SOLUTION

First, we introduce the 1-D equations governing the electron beam transport under the influence of the electrostatic force. We use the distance along the accelerator axis z as the independent variable, while the dependent electron coordinate $\zeta = z - v_0 t$ defines the particle location with respect to the bunch center, where v_0 is the velocity of the reference particle and t is the time of arrival at the transverse plane z. The particle momentum conjugate to ζ is then $\Delta\beta \equiv d\zeta/dz = 1 - \beta_0/\beta \approx (\Delta\gamma/\gamma_0)(1/\beta_0\gamma_0^2)$, where β is the particle velocity divided by the light velocity c, the Lorentz factor $\gamma \equiv 1/\sqrt{1-\beta^2}$, and $\Delta\gamma \equiv \gamma - \gamma_0$ is the difference in particle energy from the reference energy. We introduce the particle distribution function $f(\zeta, \Delta\beta; z)$ on the phase space $(\zeta, \Delta\beta)$, which satisfies

$$\frac{\partial f}{\partial z} + \Delta\beta\frac{\partial f}{\partial \zeta} - \frac{eE}{mc\beta\gamma_0^3}\frac{\partial f}{\partial \Delta\beta} = 0, \tag{1}$$

where the longitudinal component of the electric field $E(\zeta, z)$ satisfies Gauss's equation

$$\frac{\partial E}{\partial z} + \frac{\partial E}{\partial \zeta} = -\frac{e}{\varepsilon_0}\int d\Delta\beta\; f(\zeta, \Delta\beta; z). \tag{2}$$

Here, ε_0 is the vacuum dielectric constant and e is the magnitude of the electron charge. The distribution function f can be written in the Klimontovich form:

$$f(\zeta, \Delta\beta; z) = \frac{1}{A}\sum_{j=1}^{N_e}\delta[\zeta - \zeta_j(z)]\delta[\Delta\beta - \Delta\beta_j(z)], \tag{3}$$

where the sum is over all N_e eelctrons and A is the transverse area. To develop a perturbation theory, we split the distribution function into the smooth background term $f_0(\Delta\beta)$ and the rest $\hat{f}(\zeta, \Delta\beta; z)$:

$$\begin{aligned} f(\zeta, \Delta\beta; z) &= f_0(\Delta\beta) + \hat{f}(\zeta, \Delta\beta; z) \\ &\equiv n_0 g(\Delta\beta) + \hat{f}(\zeta, \Delta\beta; z), \end{aligned} \tag{4}$$

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† kwangje@aps.anl.gov
‡ lindberg@aps.anl.gov

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where, n_0 is the uniform electron beam density and the momentum distribution function g associated with f_0 is normalized such that $\int d\Delta\beta\; g(\Delta\beta) = 1$. Our solution will be simplified by introducing the Laplace transform in z and Fourier transform in ζ as follows:

$$\hat{f}_{\omega,k}(\Delta\beta) = \frac{1}{2\pi}\int_0^\infty dz\; e^{i\omega z}\int_{-\infty}^{\infty} d\zeta\; \hat{f}(\zeta,\Delta\beta;z) \quad (5)$$

$$E_{\omega,k} = \frac{1}{2\pi}\int_0^\infty dz\; e^{i\omega z}\int_{-\infty}^{\infty} d\zeta\; E(\zeta;z). \quad (6)$$

We assume that the background distribution f_0 is in some sense much larger than the perturbation $\hat{f}$, so that f_0 and $\hat{f}$ are, respectively, zeroth order and first order quantities. Additionally, we are only interested in wavelengths with $k \neq 0$, so that according to Eq. (2) the electric field is also first order ($|E| \sim |\hat{f}|$). In this case at first order we obtain the following linear system of equations:

$$(\omega - k\Delta\beta)\hat{f}_{\omega,k} + \frac{ien_0 g'}{m\beta\gamma^3}E_{\omega,k} = i\hat{f}_k(\Delta\beta;0) \quad (7)$$

$$(\omega - k)E_{\omega,k} = \frac{ie}{\varepsilon_0}\int d\Delta\beta\; \hat{f}_{\omega,k}. \quad (8)$$

In the above, $g'(\Delta\beta) \equiv dg/d\Delta\beta$, and

$$\hat{f}_k(\Delta\beta;0) = \frac{1}{2\pi}\int d\zeta\; e^{-ik\zeta}\hat{f}(\zeta,\Delta\beta;z=0)$$
$$= \frac{1}{2\pi A}\sum_{j=1}^{N_e} e^{-ik\zeta_j^0}\delta(\Delta\beta - \Delta\beta_j^0). \quad (9)$$

Here $(\zeta_j^0, \Delta\beta_j^0)$ are the initial phase space coordinates for the j^{th} particle: $(\zeta_j^0, \Delta\beta_j^0) = (\zeta_j(z=0), \Delta\beta_j(z=0))$. In the following we consider only the case $\omega \ll k$; for the relevant wavelengths we will see that $\Omega_p < k/\gamma_0^2$, so this is typically well-satisfied. Therefore, Eq. (8) can be written as

$$E_{\omega,k} = -\frac{ie}{\varepsilon_0 k}\int d\Delta\beta\; \hat{f}_{\omega,k}. \quad (10)$$

Solving the linear system Eqs. (7) and (10), we obtain

$$E_{\omega,k} = \frac{e}{2\pi k\varepsilon_0\varepsilon(k,\omega)}\frac{1}{A}\sum_{j=1}^{N_e}\frac{e^{-ik\zeta_j^0}}{\omega - k\Delta\beta_j^0}, \quad (11)$$

where we have introduced the normalized dielectric response function

$$\varepsilon(k,\omega) = 1 + \Omega_p^2\int d\Delta\beta\; \frac{g'(\Delta\beta)}{\omega - k\Delta\beta}. \quad (12)$$

Here $\Omega_p = \sqrt{e^2 n_0/\varepsilon_0 m\beta_0\gamma_0^3}$ is the electron beam plasma frequency in the laboratory frame. Having obtained the electric field, the first order Klimontovich distribution can be obtained from Eq. (7).

Since the SASE from an FEL is largely seeded by the initial density fluctuations, we consider the explicit solution for the "bunching factor" defined as

$$b_k(z) = \frac{A}{N_e}\int d\zeta d\Delta\beta\; e^{-ik\zeta}\hat{f}(\zeta,\Delta\beta;z)$$
$$= \frac{1}{N_e}\sum_{j=1}^{N_e} e^{-ik\zeta_j(z)}. \quad (13)$$

Its Laplace transform was computed in above, which can be obtained from Eqs. (10-11):

$$b_{\omega,k} = \frac{A}{N_e}\int d\Delta\beta\; \hat{f}_{\omega,k}(\Delta\beta) = \frac{ik\varepsilon_0}{eN_e}E_{\omega,k}$$
$$= \frac{i}{2\pi\varepsilon(\omega,k)}\frac{1}{N_e}\sum_{j=1}^{N_e}\frac{e^{-ik\zeta_j^0}}{\omega - k\Delta\beta_j^0}. \quad (14)$$

The inverse Laplace transformation yields

$$b_k(z) = \frac{i}{2\pi N_e}\int_L d\omega\; \frac{e^{i\omega z}}{\varepsilon(k,\omega)}\sum_{j=1}^{N_e}\frac{e^{-ik\zeta_j^0}}{\omega - k\Delta\beta_j^0}, \quad (15)$$

where the integral along the Landau contour L can be performed by finding the poles and evaluating the residues. We will find it instructive to separate the poles into two groups: the first obtained from the zeros of dielectric function defined by $\omega_q : \varepsilon(k,\omega_q) = 0$; the second are given by $\omega = k\Delta\beta_j^0$ for $j = 1, 2, \ldots, N_e$. We use the superscript C to distinguish the part of the bunching factor arising from the former set of poles with $\omega = \omega_q$, while we identify the latter poles the superscript I:

$$b_k(z) = b_k^C(z) + b_k^I(z) \quad (16)$$

with

$$b_k^C(z) = \sum_q e^{-i\omega_q}\frac{1}{\varepsilon'(k,\omega_q)}\frac{1}{N_e}\sum_{j=1}^{N_e}\frac{e^{-ik\zeta_j^0}}{\omega - k\Delta\beta_j^0} \quad (17)$$

$$b_k^I(z) = \frac{1}{N_e}\sum_{j=1}^{N_e}\frac{e^{-ik\zeta_j^0}}{\varepsilon(k,k\Delta\beta_j^0)}, \quad (18)$$

where the ω_q are the solutions of $\varepsilon(k,\omega_q) = 0$ and the prime denotes the derivative with respect to ω.

We shall see in the following that Eq. (17) and Eq. (18) correspond, respectively, to the "collective" part exhibiting plasma oscillations and to the "individual" part for which each particle moves independently. This decomposition of the bunching factor is a precise formulation of that first introduced by Pines and Bohm [1].

COLLECTIVE AND INDIVIDUAL PARTICLE COMPONENTS

In this section we illustrate the behavior of the collective component b_k^C and the single particle component b_k^I, and

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show what parameters control their relative importance. To make this calculation explicit, we will assume that the smooth part of the momentum distribution is Gaussian

$$g(\Delta\beta) = \frac{\exp[-(\Delta\beta)^2/2\sigma_{\Delta\beta}^2]}{\sqrt{2\pi}\sigma_{\Delta\beta}}, \qquad (19)$$

where $\sigma_{\Delta\beta}$ is the rms momentum width. Following standard plasma physics, the Debye length λ_D is defined as

$$\lambda_D = \sigma_{\Delta\beta}/\Omega_p. \qquad (20)$$

To study the collective part, we first consider the case $k\lambda_D \ll 1$, for which the diectric function can be approximately computed as $\varepsilon(k,\omega) \approx 1 - \Omega_p^2/\omega^2$. Thus, there are two poles $\omega_q = \pm\Omega_p$ [3], and the collective part of the bunching is

$$b_k^C(z) \approx \frac{1}{N_e}\sum_{j=1}^{N_e} e^{-ik\zeta_j^0}\left[\frac{e^{-i\Omega_p z}}{\Omega_p - k\Delta\beta_j^0} + \frac{e^{i\Omega_p z}}{\Omega_p + k\Delta\beta_j^0}\right]. \qquad (21)$$

In view our assumption $k\lambda_D \ll 1$, $\Omega_p \gg k\Delta\beta_j^0$ for most values of $\Delta\beta_j^0$, and Eq. (21) can be written as

$$b_k^C(z) = b_k(0)\cos(\Omega_p z) - \frac{ik}{\Omega_p}p_k(0)\sin(\Omega_p z). \qquad (22)$$

Here $b_k(0)$ and $p_k(0)$ are the initial values of the bunching factor and the beam energy modulation, given by

$$b_k(0) = \frac{1}{N_e}\sum_j e^{-ik\zeta_j^0} \qquad (23)$$

$$p_k(0) = \frac{1}{N_e}\sum_j \Delta\beta_j^0 e^{-ik\zeta_j^0}. \qquad (24)$$

In general, the energy modulation is given by the average

$$p_k(z) = \frac{A}{N_e}\int d\Delta\beta\, \hat{f}_k\Delta\beta = \frac{1}{N_e}\sum_j \Delta\beta e^{-ik\zeta_j}. \qquad (25)$$

By Fourier transforming (1) and integrating over $\Delta\beta$, we find that p_k and b_k are related by the continuity equation

$$p_k(z) = \frac{i}{k}\frac{\partial}{\partial z}b_k(z). \qquad (26)$$

The beam energy modulation can also be decomposed into collective and incoherent parts. When $k\lambda_D \ll 1$, the collective part of the energy modulation is

$$p_k^C(z) \approx p_k(0)\cos(\Omega_p z) - \frac{i\Omega_p}{k}b_k(0)\sin(\Omega_p z). \qquad (27)$$

Equations (22) and (27) describe the plasma oscillation.

Let us now look at the incoherent term of the bunching (18), which is given by a sum over $e^{-ik(\zeta_j^0+\Delta\beta_j^0 z)}$. This represents the phase evolution of a free particle, with the factor $1/\varepsilon(k,k\Delta\beta_j)$ interpreted as the shielding due to the other electrons. The magnitude of the incoherent term can be found from

$$\left\langle \left|b_k^I\right|^2\right\rangle = \left\langle \frac{1}{N_e}\sum_j \frac{1}{|\varepsilon(k,k\Delta\beta_j^0)|^2}\right\rangle + \left\langle \frac{1}{N_e^2}\sum_{j\neq\ell} \frac{e^{-ik[\zeta_j^0-\zeta_\ell^0+(\Delta\beta_j^0-\Delta\beta_\ell^0)z]}}{\varepsilon(k,k\Delta\beta_j^0)\varepsilon^*(k,k\Delta\beta_\ell^0)}\right\rangle, \qquad (28)$$

where the angular brackets denote taking an ensemble average. Invoking the random phase approximation, the second term above vanishes, and the first term can be written as the integral

$$\left\langle \left|b_k^I\right|^2\right\rangle = \frac{1}{N_e}\int d\Delta\beta\, \frac{g(\Delta\beta)}{|\varepsilon(k,k\Delta\beta|^2}. \qquad (29)$$

For the Maxwellian velocity distribution (19), the exact value of the integral in (29) can be found by setting up the Kramers-Kronig relation for the function $1/\varepsilon(k,\omega)$, which is analytic in the upper half ω-plane [5]. The result is [2]

$$\left\langle \left|b_k^I\right|^2\right\rangle = \frac{1}{N_e}\frac{(k\lambda_D)^2}{1+k\lambda_D^2}. \qquad (30)$$

Thus the magnitude of the incoherent part is small if $k\lambda_D \ll 1$. In that case, equipartition predicts that the electrostatic field energy density of the plasma wave modes scales as $\sim \sigma_{\Delta\beta}^2 \sim \lambda_D^2$ for each k, which implies density fluctuations $|b|^2 \sim (k\lambda_D)^2/N_e$. In the other extreme, if $k\lambda_D \gtrsim 1$ the random density fluctuations approach those of a noninteracting gas, $|b|^2 \sim 1/N_e$.

IMPLICATION FOR FELS

We are now in a position to investigate to what extent the collective plasma wave dynamics of the electron beam affect the FEL output. The effective input noise for SASE is proportional to [4]

$$\mathcal{S}_k(z_0) = \left|\sum_{j=1}^{N_e}\frac{e^{-ik\zeta_j(z_0)}}{\mu - \eta_j/\rho}\right|^2, \qquad (31)$$

where z_0 is the position of the undulator entrance, μ is the complex growth rate given by the root of the FEL dispersion relation, $\eta_j = \Delta\gamma_j/\gamma_0 = \gamma_0^2\Delta\beta_j$, and ρ is the FEL strength (Pierce) parameter. Assuming $|\mu| > |\eta_j/\rho|$, as is typically the case, we expand the SASE seed (31) keeping only the first order term:

$$\mathcal{S}_k(z_0) = N_e^2\left|b_k(z_0) + \frac{\mu}{\rho}\gamma_0^2 p_k(z_0)\right|^2. \qquad (32)$$

We note from Eqs. (23)-(24) that the magnitudes of the bunching factor and the collective momentum at $z=0$ are

$$|b_k(0)| = \frac{1}{\sqrt{N_e}} \qquad |p_k(0)| = \frac{\sigma_{\Delta\beta}}{\sqrt{N_e}}. \qquad (33)$$

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Table 1: Numerical example for the LCLS case using the data in Ref [8]. The first column lists the existing LCLS parameters at 1.5 Å, while the (hypothetical) FEL using the LCLS injector parameters set $K = 1.41$ and $\lambda_u = 7.3$ cm to obtain 1 micron radiation with the 135 MeV electron beam.

	LCLS	**LCLS injector**
Energy [GeV]	14.4 ($\gamma_0 = 28 \times 10^3$)	0.135 ($\gamma_0 = 264$)
Peak current [A]	3400	40
Normalized Energy spread (rms)	10^{-4}	2×10^{-5}
Beam size (rms) [microns]	7.7	67.3
Modulation wavelength [Å]	1.5	10^4
FEL parameter ρ	5×10^{-4}	5.5×10^{-3}
$k\lambda_D$	1.13	8.5×10^{-3}

If $k\lambda_D \lesssim 0.3$, than the collective plasma oscillation makes sense and one quarter plasma period later at $z = \pi/2\Omega_p = \Lambda_p/4$ they are given by

$$|b_k(\Lambda_p/4)| = \frac{k\lambda_D}{\sqrt{N_e}} \qquad |p_k(\Lambda_p/4)| = \frac{\Omega_p}{k\sqrt{N_e}}. \tag{34}$$

Therefore, if the undulator entrance is at $z = 0$, we have

$$\mathcal{S}_k(0) = N_e\left(1 + \left|\frac{\mu\sigma_\eta}{\rho}\right|^2\right), \tag{35}$$

where, $\sigma_\eta \equiv \gamma_0^2\sigma_{\Delta\beta}$ is the normalized rms energy spread. The second term in Eq. (35), which is due to the "momentum" noise, is in general smaller than the first term since for the FEL we require $\sigma_\eta/\rho < 1$. On the other hand, if the undulator entrance is at $z = \Lambda_p/4$ (and $k\lambda_D \leq 0.3$), the input noise becomes

$$\mathcal{S}_k(\Lambda_p/4) = N_e\left((k\lambda_D)^2 + \left|\mu\frac{\sigma_\eta}{\rho}\frac{1}{k\lambda_D}\right|^2\right). \tag{36}$$

After a quarter period of the plasma oscillation, the first term in (36) associated with the bunching is much smaller than the corresponding term in (35) when the collective behavior dominates, i.e., if $k\lambda_D \ll 1$. This is a well-known phenomenon of "shot noise reduction" after a propagation distance equal to one quarter plasma wavelength from the cathode. Recently, it was proposed in Refs. [6, 7] to employ this effect to suppress the shot noise for high-gain self-amplified spontaneous emission. However, the second term in Eq. (36), which can be attributed to the contribution from the momentum noise, can be larger than unity if $k\lambda_D \ll 1$. Furthermore, the largest possible noise reduction due to the plasma wave occurs when the two terms of (36) are equal, in which case the SASE seeding is reduced by an amount σ_η/ρ. For typical FELs under consideration we have $\sigma_\eta/\rho \sim$ 0.1-0.3.

We conclude this section by computing the role of plasma oscillations for two sets of Linac Coherent Light Source (LCLS) parameters taken from Ref. [8], that we list in Table 1. The first column shows the LCLS operating point at 1.5 Å, for which we find that $k\lambda_D > 1$ so that the collective oscillation dynamics are not relevant at these wavelengths. If we consider the beam after the injector at 135 MeV, however, then for the nominal wavelength of 1 μm we have $k\lambda_D \ll 1$, and plasma wave oscillations are important. Nevertheless, after one-quarter plasma period the reduction of the input SASE noise at this wavelength is marginal, $S_k(\Lambda_p/4) \approx N_e/2$, because of the large increase in momentum noise [the second term in (36)].

CONCLUSIONS

To properly determine the fluctuation characteristics in a highly relativistic electron beam, one must consider how discrete particle effects might modify any collective behavior. We presented a simple 1-D model that illustrates a clear division between collective and single particle behavior, showing that single particle dynamics plays a dominant role for beam fluctuations whose length scale is less than or of order the Debye length. This seems to be the regime of parameter space relevant for hard x-ray FELs. Furthermore, we have shown that even when collective behavior is important, there are important situations where a reduction in density (shot) noise due to the collective plasma wave does not lead to a marked decrease in seed power for SASE, due to the associated increase in momentum noise of the beam.

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