EFFECT OF COULOMB COLLISIONS ON ECHO-ENABLED HARMONIC GENERATION (EEHG)*

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Abstract

We calculate the EEHG bunching factor with account of the collisions and derive a simple scaling relation for the strength of the effect. Our estimates show that collisions become a limiting factor in EEGH seeding for large harmonic numbers.

INTRODUCTION

Echo-enabled harmonic generation (EEHG) for FEL seeding uses two undulator-modulators and two chicanes to introduce a fine structure into the beam longitudinal phase space which, at the end of the system, transforms into a high harmonic modulation of the beam current [1, 2]. As a result of this phase space manipulation the energy distribution function after the first chicane becomes a rapidly modulated function of energy, with the scale of the modulation of the order of the initial energy spread of the beam divided by the EEHG harmonic number. Small-angle Coulomb collisions between the particles of the beam (also known as intrabeam scattering) tend to smear out this modulation and hence to suppress the beam bunching. In this paper we calculate the EEHG bunching factor with account of the collisions and derive a simple scaling relation for the strength of the effect.

It is well known that the dominant process in Coulomb collisions is a small-angle scattering, which leads to a diffusion process in the momentum space. We first derive the diffusion coefficient for this process in the beam frame using an approximation that the longitudinal temperature of the beam is much smaller than the transverse one. We then make a Lorentz transformation into the laboratory frame, and calculate the effect of the Coulomb collisions on the bunching factor in the EEGH seeding.

LORENTZ TRANSFORMATION OF A GAUSSIAN DISTRIBUTION FUNCTION

Consider a relativistic beam with the nominal energy $E_0 = \gamma m_e c^2$ ($\gamma \gg 1$) moving along a straight path in z direction. Assuming a Gaussian distribution function of the beam in the lab frame, we write it as follows

$$f(p_x, p_y, p_z) = \frac{n_0}{(2\pi)^{3/2} p_0^2 \sigma_\theta^2 \sigma_{pz}} \times e^{-(p_x^2+p_y^2)/2p_0^2\sigma_\theta^2} e^{-(p_z-p_0)^2/2\sigma_{pz}^2}, \quad (1)$$

where n_0 is the beam density (number of particles per unit volume), p_x and p_y are the transverse components of the momenta, σ_θ is the rms angular spread of the beam, $p_0 = E_0/c$ is the nominal momentum, and σ_{pz} is the rms spread of the longitudinal momentum in the beam. The rms spreads of the transverse components of the momentum σ_{px} and σ_{py} are assumed equal to each other and are written as $p_0\sigma_\theta$, where σ_θ is the rms angular spread of the beam. Note that in the limit when $\sigma_{pz}/p_0 \gg \sigma_\theta^2$, which we will assume here (see numerical estimation at the end of this section), one can identify $c\sigma_{pz}$ with the rms energy spread in the beam σ_E[1]. In what follows we will use the relative energy $\eta = (E - E_0)/E_0$, and correspondingly the rms spread $\sigma_\eta = \sigma_E/E_0$ and replace σ_{pz} in the distribution function (1) by $p_0\sigma_\eta$.

In order to obtain the distribution function $\mathcal{F}$ in the beam frame (moving with velocity p_0/c along z) one needs to use the Lorentz transformation for the momenta, and also take into account that the particle density in the beam frame is γ times smaller than in the lab frame. In addition, we will assume that particles' velocities in the beam frame are non-relativistic. To simplify equations in what follows we will use the same notations p_x, p_y and p_z for the momenta components in the beam frame. A simple calculation gives

$$\mathcal{F}(p_x, p_y, p_z) = \frac{\mathcal{N}}{(2\pi m_e)^{3/2} T_\perp T_\parallel^{1/2}} \times e^{-(p_x^2+p_y^2)/2m_e T_\perp} e^{-p_z^2/2m_e T_\parallel}, \quad (2)$$

where $\mathcal{N} = n_0/\gamma$ is the particle density in the beam frame, $T_\perp = m_e\gamma^2\sigma_\theta^2 c^2$, $T_\parallel = m_e c^2 \sigma_\eta^2$ (here n_0, σ_θ and σ_η refer to the lab frame). The functions (1) and (2) expressed in the same variables are actually equal to each other in agreement with the fact that the distribution function is invariant with respect to the Lorentz transformation [3]. Our assumption of non-relativistic motion in the beam frame means $T_\parallel, T_\perp \ll m_e c^2$.

To illustrate the practicality of our assumptions, let us estimate the transverse and longitudinal temperatures of the beam with the normalized emittance $\epsilon = 1\,\mu$m, beam energy 1 GeV, $\gamma \approx 2000$, and the energy spread $\sigma_\eta = 10^{-4}$—typical parameters for a soft x-ray FEL beam. Assuming that the beam is transported through a beam line with the beta function of 10 m, we find the angular spread $\sigma_\theta = \sqrt{\epsilon/\gamma\beta} = 7 \times 10^{-6}$ and $T_\perp \approx 100$ eV. For the longitudinal temperature one finds $T_\parallel \approx 5 \times 10^{-3}$ eV. We see

[1] In the opposite limit, $\sigma_{pz}/p_0 \ll \sigma_\theta^2$, one cannot neglect p_x and p_y in the equation for energy, $E \approx pc \approx p_z c(1 + (p_x^2 + p_y^2)/2p_0^2)$, and the angular spread of the beam is coupled to the energy spread.

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that non-relativistic condition $T_\parallel, T_\perp \ll m_e c^2$ is well satisfied. Moreover, we also see that $T_\parallel \ll T_\perp$. We will use this condition in the next section to simplify the Coulomb collision term.

SIMPLIFIED COULOMB COLLISION TERM

We saw in the previous section, that for typical parameters of an FEL beam $\sigma_\eta \ll \gamma\sigma_\theta$, and the longitudinal temperature of the beam in the beam frame is much smaller than the transverse one with both being non-relativistic. The collision term can be simplified in this limit using an approach similar to that developed in [4].

The EEHG distribution function of the beam after the first chicane can be represented as a product of a Gaussian distribution in the transverse direction, and a rapidly modulated Gaussian over the energy [5]. We approximate this function (in the beam reference frame) by

$$\mathcal{F}(\boldsymbol{p}, t) = \mathcal{F}_2(p_x, p_y)\mathcal{F}_1(p_z, t), \tag{3}$$

where $\mathcal{F}_1$ is a one-dimensional distribution function over p_z and

$$\mathcal{F}_2(p_x, p_y) = \frac{1}{2\pi m T_\perp} e^{-(p_x^2+p_y^2)/2mT_\perp}. \tag{4}$$

The function $\mathcal{F}_1$ is normalized by $\int dp_z \mathcal{F}_1 = \mathcal{N}$. The characteristic width of $\mathcal{F}_1$ is of order of $\sqrt{m_e T_\parallel}$, and, as mentioned above, it is modulated on the scale $\sim \sqrt{m_e T_\parallel}/m$, where m is the harmonic number. The time dependence of $\mathcal{F}_1$ indicates its evolution due to Coulomb collisions, however we neglect the time variation of $\mathcal{F}_2$ because it is a much slower process compared with that of $\mathcal{F}_1$.

Due to the intrabeam scattering $\mathcal{F}$ will evolve in time and this evolution is described by the Landau collision term [6]. The collision term involves first and second derivatives of $\mathcal{F}$ with respect to momenta, of which the dominant one in our case will be $\partial^2\mathcal{F}/\partial p_z^2$, due to a rapid variation of $\mathcal{F}$ along p_z. Keeping only this term allows us to write the kinetic equation for $\mathcal{F}$ as

$$\frac{\partial \mathcal{F}}{\partial t} = \frac{1}{2}\mathcal{D}_{zz}\frac{\partial^2 \mathcal{F}}{\partial p_z^2}, \tag{5}$$

where the diffusion coefficient D_{zz} is given by

$$\mathcal{D}_{zz} = 4\pi\mathcal{N} m_e e^4 \Lambda \psi, \tag{6}$$

with

$$\psi(\boldsymbol{p}) = \int d^3p' \mathcal{F}(\boldsymbol{p}') \frac{(p_x - p'_x)^2 + (p_y - p'_y)^2}{|\boldsymbol{p} - \boldsymbol{p}'|^3}, \tag{7}$$

and Λ the Coulomb logarithm (cf. [4]). Because ψ is obtained by integration of $\mathcal{F}$, the rapid energy modulation of $\mathcal{F}$ over p_z will be averaged out, and we can use (2) in evaluation of the integral (7). In the limit $T_\parallel \ll T_\perp$ one can also approximate $|\boldsymbol{p} - \boldsymbol{p}'| \approx ((p_x - p'_x)^2 + (p_y - p'_y)^2)^{1/2}$ and carry out integration over p'_z in (7) using the normalization of $\mathcal{F}_1$:

$$\psi(p_x, p_y) = \mathcal{N}\int d^2p' \mathcal{F}_2(p'_x, p'_y) \times ((p_x - p'_x)^2 + (p_y - p'_y)^2)^{-1/2}. \tag{8}$$

Note that in our approximation ψ does not depend on p_z and is also independent of time.

To calculate the integral in (8) we use the identity $R^{-1} = \sqrt{2/\pi}\int_0^\infty d\xi e^{-\xi^2 R^2/2}$, and rewrite (8) as

$$\psi(p_x, p_y) = \sqrt{\frac{2}{\pi}}\mathcal{N}\int_0^\infty d\xi \int d^2p' \times e^{-\xi^2((p_x-p'_x)^2+(p_y-p'_y)^2)^2/2}\mathcal{F}_2(p'_x, p'_y). \tag{9}$$

The integration over p_x and p_y can now be easily carried out

$$\psi(p_x, p_y) = \frac{\mathcal{N}}{\sqrt{2\pi m T_\perp}}\int_0^\infty \frac{d\zeta}{(\zeta+1)\sqrt{\zeta}} \times e^{-(p_x^2+p_y^2)\zeta/2mT_\perp(\zeta+1)}, \tag{10}$$

where we have introduced the new integration variable $\zeta = \xi^2 m_e T_\perp$.

To obtain an equation for $\mathcal{F}_1$ we integrate (5) over p_x and p_y:

$$\frac{\partial \mathcal{F}_1}{\partial t} = \frac{1}{2}\langle\mathcal{D}_{zz}\rangle\frac{\partial^2 \mathcal{F}_1}{\partial p_z^2}, \tag{11}$$

where for $\langle\mathcal{D}_{zz}\rangle \equiv \int dp_x dp_y \mathcal{D}_{zz}\mathcal{F}_2(p_x, p_y)$ with the help of (4) and (10) one finds

$$\langle\mathcal{D}_{zz}\rangle = 4\pi\mathcal{N} m_e e^4\Lambda \int dp_x dp_y \psi(p_x, p_y)\mathcal{F}_2(p_x, p_y) = \frac{2\pi^{3/2}\mathcal{N}\sqrt{m_e}e^4\Lambda}{\sqrt{T_\perp}}. \tag{12}$$

This our result agrees with calculations of Ref. [4].

TRANSFORMATION TO THE LABORATORY FRAME

We will now express all quantities in the beam frame from the previous section in terms of the beam parameters in the lab frame. We will also average our diffusion equation over the transverse geometrical cross section of the beam.

The time in the beam frame is related to the distance s traveled in the lab frame via the transformation $t \to s/c\gamma$. The momentum p_z in the beam frame can be expressed through the energy deviation ΔE in the lab frame as $p_z \to \Delta E/\gamma c$. This converts (11) to the lab frame

$$\frac{\partial f}{\partial s} = \frac{\gamma c}{2}\langle\mathcal{D}_{zz}\rangle\frac{\partial^2 f}{\partial \Delta E^2}, \tag{13}$$

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where f is the energy distribution function in the lab frame. Recalling the definitions $\mathcal{N} = n/\gamma$, and $T_\perp = m\gamma^2\sigma_\theta^2 c^2$ and introducing the diffusion coefficient D in the lab frame as

$$D = \gamma c \langle \mathcal{D}_{zz} \rangle = \frac{2\pi^{3/2} n e^4 \Lambda}{\gamma \sigma_\theta}, \tag{14}$$

we arrive at

$$\frac{\partial f}{\partial s} = \frac{1}{2} D \frac{\partial^2 f}{\partial \Delta E^2}. \tag{15}$$

In our derivation we tacitly assumed that the beam density n is a constant. In reality, it is a function of x, y and z which makes D and f also dependent on these coordinates. Since in EEHG we are interested in one-dimensional dynamics in z direction we will average (15) over the transverse cross section of the beam assuming that the dependence of n and f versus x and y is given by $\exp[-(x^2 + y^2)/2\sigma_\perp^2]$ with $\sigma_\perp$ the rms transverse size of the beam. The result of the integration does not change the functional form of equation (15), but replaces the beam density n in (14) by the following expression

$$n \to \frac{1}{4\pi\sigma_\perp^2} \frac{I}{r_e I_A}, \tag{16}$$

where I is the beam current, $I_A = mc^3/e \approx 17$ kA is the Alfven current and r_e is the classical electron radius. After the averaging the distribution function $f = f(z, \Delta E, s)$ remains dependent on ΔE, s and z. Finally, using the normalized transverse emittance $\varepsilon = \gamma\sigma_\theta\sigma_\perp$, we obtain for the diffusion coefficient

$$D = \frac{\pi^{1/2} e^4 \Lambda}{2\varepsilon r_e \sigma_\perp} \frac{I}{I_A}. \tag{17}$$

In practical units, assuming $\Lambda \approx 8$ [7]

$$D = 3.1 \frac{I\ [\mathrm{kA}]}{(\epsilon_x\ [\mu\mathrm{m}])(\sigma_x\ [100\,\mu])} \frac{\mathrm{keV}^2}{\mathrm{m}}. \tag{18}$$

APPLICATION FOR EEHG

In application to EEHG we will adopt a model, in which we take collisions as occurring in a drift section of length l after the first chicane. The justification of this approach lies in the fact that in this region the distribution function, being modulated in energy, is most sensitive to the collisions. This energy modulation actually persists through the second undulator and the second chicane, where it is transformed into a high-harmonic density modulation of the beam. While a more accurate theory should properly treat the collision processes inside the second modulator and the second chicane, in our model we ignore them. For a pessimistic estimate on can add their lengths to l with the assumption that the beam transverse size remains constant throughout the system, and use for l the combined lengths of the drift after the first chicane, the second undulator and the second chicane.

In what follows, we use notations of Ref. [2] with $A_1 = \Delta E_1/\sigma_E$ and $A_2 = \Delta E_2/\sigma_E$ for dimensionless amplitudes of the energy modulation of the beam, $B_1 = R_{56}^{(1)} k_1 \sigma_E/E_0$ and $B_2 = R_{56}^{(2)} k_1 \sigma_E/E_0$ for the dimensionless strengths of chicanes, $p = (E - E_0)/\sigma_E$ as the dimensionless energy deviation variable (not to be confused with momentum used in the previous sections) and $\zeta = k_L z$ as a longitudinal coordinate in the beam, with k_L the wave number of the laser (assumed equal in both modulators). The distribution function after the first chicane is given by the following equation (see [2])

$$f_1(\zeta, p) = \frac{1}{\sqrt{2\pi}} e^{-(p - A_1 \sin(\zeta - B_1 p))^2/2}. \tag{19}$$

Plot of this function for $A_1 = 3$ and $B_1 = 8.47$ (the value of B_1 is an optimized value for the 50th harmonic EEHG) is shown in Fig. 1. In order to solve the diffusion equation (15) for f we make the Fourier transformation over the energy variable

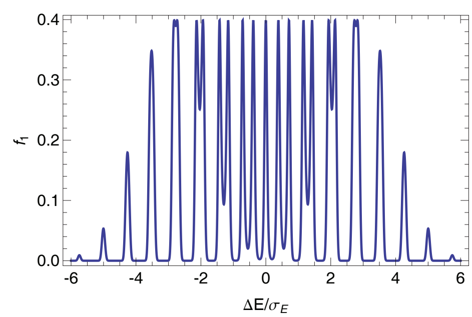

Figure 1: Distribution function $f_1(0, \Delta E/\sigma_E)$ optimized for the 50th harmonic as a function of the normalized energy deviation.

$$\hat{f}(\zeta, \mu, s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dp e^{ip\mu} f(\zeta, p, s). \tag{20}$$

The Fourier transformed Eq. (15)

$$\frac{\partial \hat{f}}{\partial s} = -\frac{1}{2\sigma_E^2} D\mu^2 \hat{f} \tag{21}$$

can easily be solved

$$\hat{f}(s, \zeta, \mu) = \hat{f}_1(\zeta, \mu) e^{-Ds\mu^2/2\sigma_E^2}. \tag{22}$$

Denoting the distribution function after the drift l, at the entrance to the second modulator, by f_2 we find

$$f_2(\zeta, p) = \int_{-\infty}^{\infty} d\mu e^{-ip\mu} \hat{f}_1(\zeta, \mu) e^{-S\mu^2/2}, \tag{23}$$

with $S = Dl/\sigma_E^2$. The second stage of EEHG carries out the following transformation (see [2])

$$p' = p + A_2 \sin(\zeta), \quad \zeta' = \zeta + B_2 p + B_2 A_2 \sin(\zeta), \tag{24}$$

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which transforms $f_2(\zeta, p)$ into a new distribution function $f_3(\zeta', p')$. Finally, the bunching factor of the mth harmonic can be computed as a half of the Fourier harmonic of $f_3(\zeta', p')$ integrated over energy:

$$b_m = \frac{1}{2}\frac{1}{2\pi}\int_0^{2\pi} d\zeta' \int_{-\infty}^{\infty} dp' e^{im\zeta'} f_3(\zeta', p'). \quad (25)$$

Due to the symplecticity of the transformation (24) we can replace integration over ζ' and p' in (25) by integration over ζ, p variables which also reverts f_3 to f_2:

$$b_m = \frac{1}{2}\frac{1}{2\pi}\int_0^{2\pi} d\zeta \int_{-\infty}^{\infty} dp e^{im\zeta'(\zeta,p)} f_2(\zeta, p) \quad (26)$$
$$= \frac{1}{2}\frac{1}{2\pi}\int_0^{2\pi} d\zeta \int_{-\infty}^{\infty} dp e^{im(\zeta + B_2 p + B_2 A_2 \sin(\zeta))} f_2(\zeta, p).$$

Using (23) we can carry out integration over p and μ (the integration over p results in the delta function $\delta(\mu - mB_2)$ which makes the integration over μ trivial) and express the result in terms of the function $\hat{f}_1$

$$b_m = \frac{1}{2} e^{-Sm^2 B_2^2/2} \int_0^{2\pi} d\zeta e^{im(\zeta + B_2 A_2 \sin(\zeta))} \hat{f}_1(\zeta, mB_2). \quad (27)$$

We see that the bunching factor with Coulomb collisions can be written as a product of the bunching factor without collisions $b_m^{(0)}$ and an exponential suppression factor $e^{-Sm^2 B_2^2/2}$:

$$b_m = b_m^{(0)} e^{-l/L}, \quad (28)$$

where $L = 2\sigma_E^2 / Dm^2 B_2^2$.

NUMERICAL EXAMPLES AND CONCLUSION

For illustration we consider a soft x-ray EEHG FEL scheme with emittance $\epsilon = 1\ \mu$m, beam peak current of 1 kA, and the rms energy spread 100 keV. We also assume the rms transverse bunch size of 100 μm. Eq. (18) then gives $D = 3.1$ keV^2/m. We considered three EEHG scenarios with the harmonic number $m = 50$, 100 and 200. For all 3 cases we assumed that the dimensionless modulation amplitude were $A_1 = 3$ and $A_2 = 6$. The optimized values of B_1 and B_2, the bunching factors without collisions, and the decay distance L are shown in Table 1. The exponential decay with the three values of L from Table 1 are shown in Fig. 2. Figure 3 shows smearing out of the distribution function for 50th harmonic after the distance $s = 0.4L$.

Table 1: EEHG Parameters and the Decay Lengths L for Three Scenarios

m	A_1	A_2	B_1	B_2	$b_m^{(0)}$	L (m)
50	3	6	4.7	0.18	0.088	80
100	3	6	16.9	0.17	0.071	22.5
200	3	6	34.3	0.17	0.047	5.6

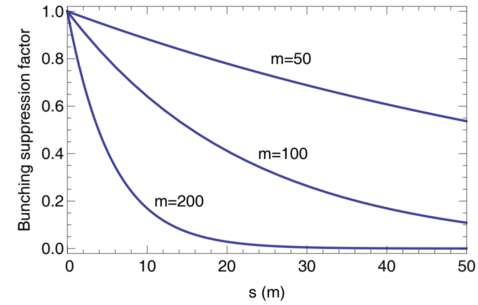

Figure 2: Plot of functions $e^{-s/L}$ for the three values of L from Table 1.

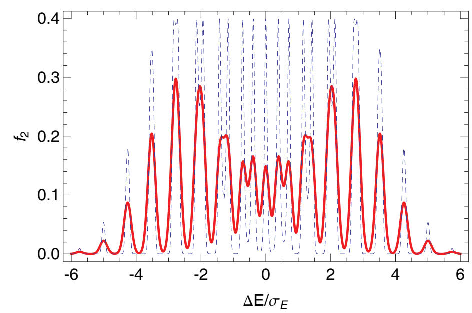

Figure 3: Blue dashed line: the distribution function from Fig. 1; red line: the same distribution function evolved due to Coulomb collisions after the distance $s = 0.4L$.

As one can see from these results, Coulomb collisions represent a serious limiting factor for the EEHG seeding in the range of harmonic numbers exceeding 10^2.

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