EFFECT OF ENERGY SPREAD ON START CURRENT OF SMITH-PURCELL BWO *

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Abstract

We perform a linear analysis of Maxwell-Vlasov equation for Smith-Purcell Backward Wave Oscillator, including the energy spread of the initial beam distribution. We use this analysis to study the dependence of start current on the energy spread of the initial electron beam distribution. The effect of beam emittance is also included through the equivalent energy spread. Results of linear analysis are compared with full nonlinear numerical simulations.

INTRODUCTION

Smith-Purcell (SP) free-electron laser in the terahertz (THz) regime using a low energy electron beam is a backward wave oscillator (BWO) [1,2]. The effect of energy spread on the performance of a BWO having corrugated wall waveguide structure have been performed numerically [3] as well as as analytically [4]. However, previous analyses of SP-BWO [1,2,5,6] have ignored the effect of finite energy spread in the electron beam. The effect of finite energy spread is to increase the start current, which is defined as the minimum electron beam current required for coherent electromagnetic oscillations to grow in the BWO. In addition to this, the energy spread also reduces the saturated power level attained in SP-BWO. In this paper, we present an analysis of the effect of energy-spread on the performance of SP-BWO using Maxwell-Vlasov equations in the linear regime. The results of linear analysis are compared with nonlinear simulation results. Using numerical simulations, we also study the effect of energy spread on saturated power.

In the next section, we present the linear analysis and use it to study the effect of energy spread on start current. Results of analytic calculation and comparison with one-dimensional numerical simulations are discussed in the following section. The equivalent energy spread arising due to finite emittance and its effect on the start current is then discussed and finally, conclusions are presented in the last section.

LINEAR ANALYSIS

We start with a brief description of the SP-BWO system. The schematic of a SP-BWO is shown in Fig. 1 along with the coordinate system. As shown in Fig. 1, a sheet electron beam propagates at a height d from the top surface of a reflection grating, with a speed βc along z-axis, where c is the speed of light. The grating has grooves of width w and depth h and extends uniformly to the positive and negative x-direction and we assume the system to have translational invariance in x-direction. The length of the grating and its period are L and λ_g respectively. In a SP-BWO,

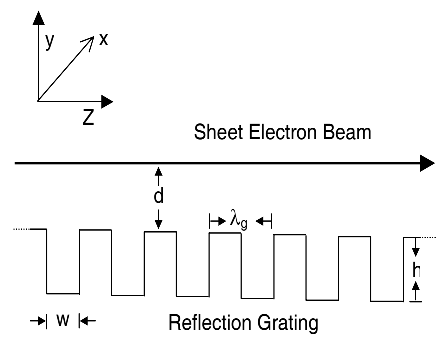

Figure 1: Schematic of a Smith-Purcell BWO.

the electron beam interacts with the co-propagating surface electromagnetic mode supported by the grating. The co-propagating surface mode has a group velocity in the direction opposite to the electron beam for low electron beam energy. The evolution of the backward surface mode due to interaction with the co-propagating electron beam can be described using following Maxwell-Lorentz equations [2].

$$\frac{\partial \mathcal{E}}{\partial \tau} - \frac{\partial \mathcal{E}}{\partial \zeta} = -\mathcal{J}\langle e^{-i\psi}\rangle, \quad (1)$$

$$\frac{\partial \eta_i}{\partial \zeta} = (\mathcal{E} + \mathcal{E}_{sc})e^{i\psi_i} + c.c., \quad (2)$$

$$\frac{\partial \psi_i}{\partial \zeta} = \eta_i, \quad (3)$$

$$\mathcal{E}_{sc} = iQ\langle e^{-i\psi}\rangle, \quad (4)$$

where, the notations used are described in Ref. [2]. Here, $\mathcal{E}$ is the dimensionless complex electric field of the surface mode, $\mathcal{E}_{sc}$ is the dimensionless complex space charge field, $\mathcal{J}$ is the dimensionless beam current, $Q = (\mathcal{J}/\chi L)(\chi_1 - e^{2\Gamma_0 d})$ is the space charge parameter, $\Gamma_0 = k_0/\gamma$, k_0 is the z-component of the wave-vector of the surface mode, γ is electron energy in units of its rest mass energy, τ is the dimensionless time, $\zeta = z/L$ is the normalized distance along the grating, ψ_i is the phase of i^{th} electron, η_i is the dimensionless relative energy of the i^{th} electron and χ and χ_1 are related to the singularity associated with the surface mode as defined in Ref. [2].

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The nonlinear analysis of the interaction of electron beam with the surface mode in a SP-BWO can be done using the above set of equations and solving them numerically [2]. For a given τ, the dynamical variables ψ_i and η_i of electrons are propagated in ζ using Eqs. (2,3) provided $\mathcal{E}$ and $\mathcal{E}_{sc}$ are known functions of ζ. We then propagate $\mathcal{E}$ and $\mathcal{E}_{sc}$ in τ using Eqs. (1,4). The evolution of the system is thus understood by solving these equations self consistently.

The linear analysis can be performed by writing down the linearized Vlasov equation coupled with Maxwell equation. We define the distribution function $f(\psi, \eta, \zeta, \tau)$ as proportional to the number of electrons per unit interval in ψ and η at given location ζ at time τ. Note that the normalization condition of the above distribution functions is given by

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{0}^{2\pi} f(\psi, \eta, \zeta, \tau) d\psi d\eta = 1. \quad (5)$$

The Vlasov equation for the distribution function, using Eqs. (2-3) can be written as

$$\frac{\partial f}{\partial \zeta} + \eta\frac{\partial f}{\partial \psi} + [(\mathcal{E} + \mathcal{E}_{sc})e^{i\psi_i} + c.c.]\frac{\partial f}{\partial \eta} = 0. \quad (6)$$

For a given τ, the distribution function is propagated in ζ using the above Vlasov equation. The amplitude of electric field in surface mode and the space-charge field is propagated in τ using Eqs. (1,3), which are Maxwell equations. The coupled Maxwell-Vlasov equtions should be solved self-consistently to find out the evolution of the system. This approach is more suitable for performing analytical calculations if we want to include the effect of energy spread.

We can express the distribution function as a sum of unperturbed part and the perturbed part as follows

$$f(\psi, \eta, \zeta, \tau) = f_0(\psi, \eta, \zeta) + \delta f(\psi, \eta, \zeta)e^{\nu\tau}, \quad (7)$$

where the unperturbed part $f_0(\psi, \eta, \zeta)$ is the distribution function in absence of any electromagnetic field. Here, we have assumed $e^{\nu\tau}$ type time dependence for the perturbed part in order to perform linear analysis. We use this time dependence for $\mathcal{E}$ and $\mathcal{E}_{sc}$ as well. Similarly, we can assume $e^{\kappa\zeta}$ type space dependence for $\mathcal{E}$, $\mathcal{E}_{sc}$ and δf. Doing this, we obtain the following form of linearized Vlasov equation

$$\frac{\partial \delta f}{\partial \zeta} + \eta\frac{\partial \delta f}{\partial \psi} + [(1 - \frac{iQ}{\mathcal{J}}(\nu - \kappa))\mathcal{E}e^{i\psi} + c.c.]\frac{\partial f_0}{\partial \eta} = 0. \quad (8)$$

We then multiply the above equation with $e^{-i\psi}$ and integrate with respect to ψ and obtain

$$\frac{1}{2\pi}\int_{0}^{2\pi} \delta f e^{-i\psi} d\psi = [i\frac{Q}{\mathcal{J}}(\nu - \kappa) - 1]\frac{\mathcal{E}}{(\kappa + i\eta)}\frac{\partial f_0}{\partial \eta}. \quad (9)$$

On the other hand, putting $e^{\nu\tau}$ type time dependence and $e^{\kappa\zeta}$ type space dependence for $\mathcal{E}$ in Eq. (1), we obtain the following equation

$$(\nu - \kappa)\mathcal{E} = -\mathcal{J}\frac{1}{2\pi}\int_{0}^{2\pi} \delta f e^{-i\psi} d\psi d\eta. \quad (10)$$

Combining Eq. (9) and (10), we obtain the following dispersion relation

$$(\nu - \kappa) = [i\mathcal{J} + Q(\nu - \kappa)]\int \frac{1}{(\kappa + i\eta)^2} f_0(\eta) d\eta \quad (11)$$

Above is the generalized form of the dispersion relation obtained previously [2]. Note that for monoenergetic electron beam, if we put $f_0(\eta) = \delta(\eta)$, we get back the result obtained in Ref. [2]. By putting the given initial energy distribution of the electron beam, we can evaluate the integral on the right side of the above equation and thus introduce the effect of finite energy spread. We assume a step function energy distribution given by $f_0(\eta) = 1/2\Delta\eta$ for $-\Delta\eta < \eta < +\Delta\eta$ and $f_0(\eta) = 0$, otherwise. For this energy distribution, we obtain the following dispersion relation, which is cubic in κ

$$(\kappa^2 + \Delta\eta^2 - Q)(\nu - \kappa) = i\mathcal{J}. \quad (12)$$

Note that $\Delta\eta$ is related to energy spread $\Delta\gamma mc^2$ by the relation $\Delta\eta = k_0 L\Delta\gamma/\beta^2\gamma^3$. Here, mc^2 is the rest mass energy of the electron. For a given value of ν, the above cubic equation gives three possible solution κ_1, κ_2 and κ_3 for κ. Hence, the more general solution for evolution of $\mathcal{E}$ has the form

$$\mathcal{E} = e^{\nu\tau}[A_1 e^{\kappa_1\zeta} + A_2 e^{\kappa_2\zeta} + A_3 e^{\kappa_3\zeta}]. \quad (13)$$

The possible values of ν are obtained by satisfying the boundary conditions, which are:

$$\mathcal{E}\big|_{\zeta=1} = 0, \quad (14)$$

$$\int\int e^{-i\psi}\delta f d\psi d\eta\big|_{\zeta=0} = 0, \quad (15)$$

$$\frac{\partial}{\partial \zeta}\int\int e^{-i\psi}\delta f d\psi d\eta\big|_{\zeta=0} = 0. \quad (16)$$

These boundary conditions are to be satisfied for $\tau \geq 0$. The first boundary condition here implies that there is no input field in the surface mode at the exit of the grating ($\zeta = 1$). Note that in a BWO, the surface mode gets amplified in the direction opposite to that of the electron beam and hence, the input electric field in the surface mode needs to be specified at the exit of the grating. The second boundary condition implies that there is no microbunching in the electron beam at the entrance to the grating ($\zeta = 0$). The third boundary condition implies that there is no energy-phase correlation in the electron beam at the grating entrance. These boundary conditions lead to the following condition

$$(\kappa_1^2 - Q)(\kappa_2 - \kappa_3)e^{\kappa_1} + (\kappa_2^2 - Q)(\kappa_3 - \kappa_1)e^{\kappa_2} + (\kappa_3^2 - Q)(\kappa_1 - \kappa_2)e^{\kappa_3} = 0. \quad (17)$$

The above equation is same as obtained earlier in Ref. [2] for the case of mono-energetic electron beam. The effect of the finite energy-spread is seen only in the dispersion relation (Eq. (12)). For a given $\mathcal{J}$, Q and $\Delta\eta$, the above equation is a transcendental equation in ν since κ_1, κ_2 and κ_3 are functions of ν through Eq. (12). One can then solve the above equation numerically. We find that there exists a threshold value $\mathcal{J}_s$ above which the real part of ν is positive. This dimensionless start current $\mathcal{J}_s$ is a function of $\Delta\eta$.

In our analysis so far, we have not taken the effect of attenuation due to finite conductivity of grating material and also the reflection at the two ends. Maxwell-Lorentz equation for the case where these effects are included are discussed in Ref. [6]. Using these modified Maxwell-Lorentz equations and following the approach discussed in this paper, we can easily show that Eq. (12) and (17) get generalized to the following form

$$(\kappa^2 + \Delta\eta^2 - Q)(\nu + \alpha L - \kappa) = i\mathcal{J}. \tag{18}$$

$$\begin{aligned}\mathcal{R}e^{-\alpha L}e^{-i(2k_0L - i\frac{\nu}{d_1})}[(\kappa_1^2 - Q)(\kappa_2 - \kappa_3)\\ +(\kappa_2^2 - Q)(\kappa_3 - \kappa_1) + (\kappa_3^2 - Q)(\kappa_1 - \kappa_2)]\\ +(\kappa_1^2 - Q)(\kappa_2 - \kappa_3)e^{\kappa_1} + (\kappa_2^2 - Q)(\kappa_3 - \kappa_1)e^{\kappa_2}\\ +(\kappa_3^2 - Q)(\kappa_1 - \kappa_2)e^{\kappa_3} = 0,\end{aligned} \tag{19}$$

where α is the attenuation coefficient of the surface mode due to finite conductivity of grating material [5], $d_1 = (\beta + \beta_g)/(\beta - \beta_g)$, β_g is the group velocity of the surface mode in units of c, $\mathcal{R} = -\rho_0\rho_L$, ρ_0 and ρ_L are the complex reflection coefficient at the entrance and exit of the grating respectively [6].

The above two equation can be used to find out the dimensionless start current $\mathcal{J}_s$ as a function of Q, $\Delta\eta$, αL and $\mathcal{R}$.

RESULTS AND DISCUSSIONS

Using the results of the previous section, we calculate the dimensionless start current as a function of energy spread as shown in Fig. 2. This is an universal curve valid for SP-BWO, independent of any parameter. Note that here we have taken $Q = 0$, $\alpha = 0$ and $\mathcal{R} = 0$. We find that as the energy spread increases, the start current increases. For different values of αL, we can get a family of curves. Next, we present the comparison of the result of analytic calculation with that of the numerical simulation. For numerical simulation, we have extended our earlier computer code for SP-BWO [2] to include the energy spread in the electron beam. We used the parameters used in Ref. [7], which are $(\gamma - 1)mc^2$ = 35 keV, λ_g = 173 μm, h = 130 μm, w = 110 μm, $\lambda = 2\pi/\beta k_0$ = 761 μm, αL = 0.59(1 - i), d = 22.6 μm and Δx = 5.00 mm. Here, Δx is the width of the sheet beam in x direction. For these parameters, $\mathcal{J}_s$ is obtained to be 11.1 for $\Delta\eta$ = 2.0. The relationship between the dimensionless start current and the start current is given by [2]

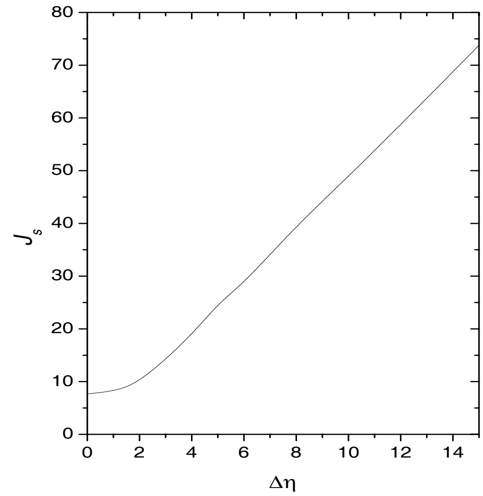

Figure 2: Dimensionless start current $\mathcal{J}_s$ as a function of energy spread $\Delta\eta$, for $Q = 0, \alpha L = 0, \mathcal{R} = 0$.

$$I_s = \mathcal{J}_s \frac{I_A \Delta x}{4\chi} \frac{\beta^3\gamma^4}{k_0 L^3} e^{2\Gamma_0 d}, \tag{20}$$

where I_A = 17 kA is the Alfvén current. Hence, $\mathcal{J}_s$ = 11.1 corresponds to a start current of 38 mA for these parameters. Note that $\Delta\eta = 2$ implies that rms energy spread $\Delta\gamma mc^2$ = 346 eV, which means that the relative energy spread is $\sim 1\%$. Note that for DC electron guns, an energy spread much smaller than this can be produced. But, for a pulsed gun, an energy spread of 1% could be a typical value. Fig. 3 shows the evolution of power for different beam currents for the above parameters. We find that the power starts building up after a current of 40 mA. This is in good comparison with the start current of 38 mA as calculated from our analytic calculation. We have also studied the effect of energy spread on the saturated behavior of SP-BWO. Fig. 4 shows the evolution of power in the surface mode for different values of energy spread. We find that as the energy spread increases, the saturated power decreases. Note that for the parameters chosen, the operating current of 45 mA is close to the start current of 40 mA for $\Delta\eta$ = 2.0. Hence, as $\Delta\eta$ approaches 2.0, the saturated power becomes very small.

It is important to point out that if the electron beam has a finite normalized rms emittance ϵ_n and is focused to a rms beam size σ, then the electrons would have a spread in transverse momenta as well as in longitudinal momenta. The spread in the longitudinal component of velocity can be calculated as follows. If electrons have rms spread θ in divergence, their longitudinal velocity v_z will have a spread $v_z\theta^2/2$. This gives rise to an rms energy spread $\Delta\gamma mc^2$, where $\Delta\gamma$ is given by

$$\Delta\gamma = \frac{\gamma\epsilon_n^2}{2\sigma^2}, \tag{21}$$

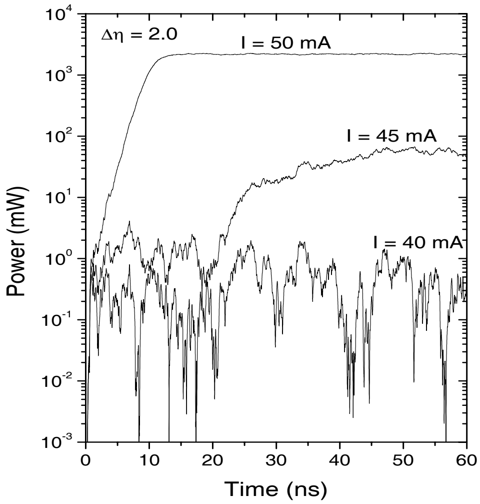

Figure 3: Evolution of power in the surface mode in SP-BWO for different values of beam current as calculated by numerical simulation. Parameters used in the numerical simulation are described in the text.

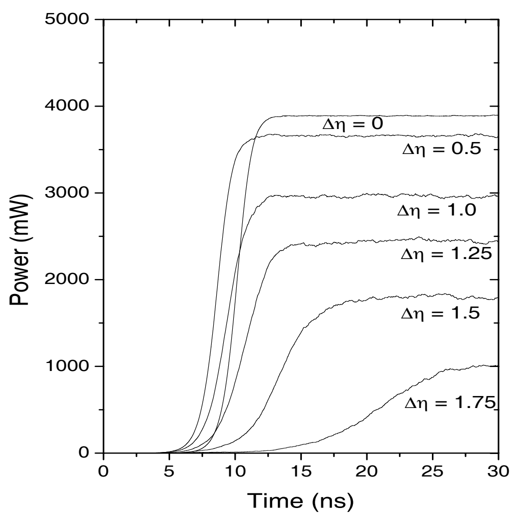

Figure 4: Evolution of power in the surface mode in SP-BWO for different values of $\Delta\eta$ as calculated by numerical simulation. Beam current is taken to be 45 mA. Other parameters used in the numerical simulation are described in the text.

which can be converted to $\Delta\eta$ using the formula described in the previous section. We can use our analysis to find out the value of emittance that can be tolerated such that the start current and the saturated power does not get effected significantly. Here, we apply the above formula in the vertical direction. For the example discussed earlier in this Section, σ_y = 11.3 μm. Applying the above formula, and using the result that $\Delta\eta$ = 1.0 can be easily tolerated as seen from Fig. 4, we find that we can in principle tolerate ϵ_{ny} up to 2.8 $\times 10^{-7}$ m-rad , provided that required field can be produced to focus the beam to the vertical rms size σ_y = 11.3 μm with this emittance.

Note that in this paper, we have considered only the step function initial energy distribution for simplicity in the calculation. It is possible to do a general analysis for an arbitrary symmetric energy distribution, using asymptotic series expansion for integral in Eq. (11) as described in Ref. [4].

CONCLUSIONS

We have shown that using linearized Maxwell-Vlasov equations, we can calculate analytically the start current in an SP-BWO as a function of energy spread and other parameters like attenuation due to finite conductivity, reflection at grating ends and the space charge. The analytic calculation has been found to compare well with the results of numerical simulation. The saturated power as a function of energy spread has been studied using these numerical simulations. We have also discussed the equivalent energy spread arising due to finite beam emittance and focusing, using which we can find out the effect of emittance on the performance of SP-BWO.

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