Transverse-Longitudinal Coupling in High Power Cyclotrons

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1. "Traditional" strategy to operate isochronous high intensity cyclotrons.
2. Space charge dominated acceleration (PSI Injector II).
3. Simplified model and the influence of the phase $\phi(E)$.
4. Conditions for space charge induced "longitudinal focusing".
5. Linear Model versus OPAL simulations.
6. Additional requirements?!
7. What's "negative mass instability" anyway?
8. Conclusions.

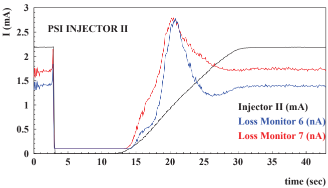
Space Charge “Dominated” Acceleration

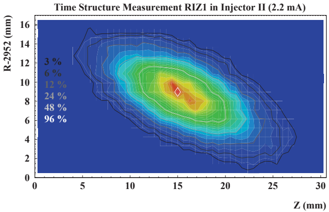
Consider an isochronous cyclotron with space charge.
“Naive” expectations:

- No longitudinal focusing (isochronism).
- Longitudinal space charge (SC) increases phase width.
- Energy gain depends on phase ⇒ increase energy width (i.e. momentum spread).
- Large momentum spread ⇒ large beam width ⇒ high losses at extraction.

Countermeasures:

1. Flattop cavity to increase phase acceptance.
2. Increase cavity voltage: less turns ⇒ lower losses (Joho's N^3-law [1].





Counterfacts:

- PSI Injector II with 2.4 mA without flattop and low losses.
- Explanation: Space charge "dominated" acceleration.
- Two bunchers in front of cyclotron (increase SC forces).
- Injector II has rel. high ν_r and ν_z (increase SC forces).
- Max. losses at intermediate beam current (see plot).
- Extremely contra-intuitive. And it works.
- But: What is it and how does it work?

$\Rightarrow$ Develop simple model:

- Transverse - longitudinal only ($\Rightarrow$ sectors can be omitted.)
- $\Rightarrow$ Use rotational symmetrie: $\vec{B} = \vec{e}_z\, B_0\, \gamma$.
- $\Rightarrow$ The (matched) beam sizes are constant.
- $\Rightarrow$ Space charge forces are constant.
- $\Rightarrow$ Linear approximation for SC forces.
- $\Rightarrow$ EQOM should have a simple solution.
- Use TRANSPORT like description in local coordinates: (horiz./vert./long./)=(x,y,z).
- Assume coasting beam, no acceleration.

First-Order Differential Equation

Single particle dynamics:

- Radial coordinate $x = r(\theta) - r_0$ and x'.
- Longitudinal position $z = r_0\,(\theta - \theta_0)$.
- Momentum deviation $\delta = \frac{\Delta p}{p_0}$.
- Put in state vector $\psi = (x, x', z, \delta)^T$ in local co-moving curvilinear coordinates.
- Define $h = 1/r_0$ as curvature of orbit.

First-Order Differential Equation

The linearized EQOM including space charge are:

$$\dot{\psi} = \mathbf{F}\,\psi\,,$$

with 4×4 "Hamiltonian" matrix **F**.
The solution (for constant **F**) is

$$\psi(s) = \exp\left(\mathbf{F}\,s\right)\psi(0) = \mathbf{M}(s)\,\psi(0)\,.$$

Note: Without space charge, **M** is the symplectic transfer matrix of a bending magnet.

Force Contributions: RF

Explicitely:

$$\frac{d}{ds}\begin{pmatrix} x \\ x' \\ z \\ \delta \end{pmatrix} = \begin{pmatrix} . & 1 & . & . \\ -k_x + K_x & . & . & h \\ -h & . & . & \frac{1}{\gamma^2} \\ . & . & K_z\,\gamma^2 + K_{rf} & . \end{pmatrix}\begin{pmatrix} x \\ x' \\ z \\ \delta \end{pmatrix},$$

Focusing terms and defocusing terms (SC) are colored. Dispersive coupling term $h = 1/r$. Drift terms in black, linearized RF term in orange (debunching for $K_{rf} > 0$).

$$K_{rf} \approx \frac{q\,V_0\,\sin(\phi)}{m\,c^2\,\gamma\,\beta^2}\,\frac{h^2\,N_h}{2\,\pi}\,.$$

But if $\beta \simeq 1$, then factor in K_{rf} is of order $q\,V_0/m\,c^2 \simeq 10^{-3} \ll 1$. Additionally we have $\sin\phi \approx 0 \Rightarrow$ neglect RF-term!

$$\mathbf{F} = \begin{pmatrix} \cdot & 1 & \cdot & \cdot \\ -k_x + K_x & \cdot & \cdot & h \\ -h & \cdot & \cdot & \frac{1}{\gamma^2} \\ \cdot & \cdot & K_z\,\gamma^2 & \cdot \end{pmatrix}.$$

K_x and K_z represent horizontal and longitudinal space charge forces [2]:

$$\begin{aligned} K_x &= \frac{K_3\,(1-f)}{(\sigma_x+\sigma_y)\,\sigma_x\,\sigma_z} \\ K_z &= \frac{K_3\,f}{\sigma_x\,\sigma_y\,\sigma_z} \\ K_y &= \frac{K_3\,(1-f)}{(\sigma_x+\sigma_y)\,\sigma_y\,\sigma_z} \end{aligned}$$

$$\begin{aligned} K_3 &= \frac{3\,q\,I\,\lambda}{20\,\sqrt{5}\,\pi\,\varepsilon_0\,m\,c^3\,\beta^2\,\gamma^3} \\ f &\approx \frac{\sqrt{\sigma_x\,\sigma_y}}{3\,\gamma\,\sigma_z} \end{aligned}$$

Note that always

$$k_x \gg K_x \quad > \quad 0 \qquad\qquad k_x \gg K_z \quad > \quad 0\,.$$

Focusing means that **F** has imaginary eigenvalues.
The eigenvalues of **F** ($\pm i\,\Omega_+$ and $\pm i\,\Omega_-$) are:

$$
\begin{aligned}
a &\equiv \tfrac{k_x - K_x - K_z}{2} \\
b &\equiv K_z\,(K_x + h^2\,\gamma^2 - k_x) \\
\Omega_+ &= \sqrt{a + \sqrt{a^2 - b}} \\
\Omega_- &= \sqrt{a - \sqrt{a^2 - b}}\,.
\end{aligned}
$$

- If b is negative $\Rightarrow a < \sqrt{a^2 - b}$
- $\Rightarrow \Omega_-$ imaginary
- $\Rightarrow$ solution is divergent (i.e. unstable).
- $\Rightarrow$ b must be positive to give real–valued frequencies.

With $b \ll a$, $K_x \ll k_x$ and $K_z \ll k_x$ and assumption of perfect isochronism: $k_x = h^2\,\gamma^2 = h^2\,\nu_r^2$, we approximate $a \approx \frac{k_x}{2}$ and $b \approx K_x\,K_z$:

$$\Omega_+ = \sqrt{a + \sqrt{a^2 - b}} \approx h\,\nu_r\,\left(1 - \frac{K_x\,K_z}{k_x^2} - \dots\right)$$

$\Rightarrow$ Ω_+ is horizontal focusing, reduced by space charge.

$$\Omega_- = \sqrt{a - \sqrt{a^2 - b}} \approx \sqrt{\frac{K_x\,K_z}{2}}\,\left(1 + \frac{K_x\,K_z}{2\,k_x^2} + \dots\right)\,.$$

$\Rightarrow$ Ω_- is effective longitudinal focusing, induced by space charge and coupling.

Focusing Condition

Focusing requires

$$\begin{aligned} b &= K_z\,(K_x + h^2\,\gamma^2 - k_x) > 0 \\ \Rightarrow & \quad K_x > k_x - h^2\,\gamma^2 \end{aligned}$$

The radial focusing force k_x is given by:

$$k_x = h^2\,(1+n) = h^2\,\left(1 + \frac{r}{B}\,\frac{dB}{dr}\right)$$

The isochronous field plus a small but important field error ε:

$$B(r) = B_0\,\gamma\,(1+\varepsilon) = B_0\,\frac{1+\varepsilon}{\sqrt{1-(r/a)^2}}\,,$$

This gives

$$k_x = h^2\,\gamma^2 + \frac{1}{r}\,\frac{d\varepsilon}{dr}\,.$$

Focusing condition:

$$K_x > \frac{1}{r}\,\frac{d\varepsilon}{dr}$$

$\omega_0 = N_h\,\omega_{rf}$ is nominal orbital frequency, N_h is the harmonic number, ω real orbital frequency and ϕ is phase. Then:

$$\varepsilon \approx 1 - \frac{\omega_0}{\omega} = -\frac{1}{2\,\pi\,N_h}\,\frac{d\phi}{dE}\,\frac{dE}{dn}\,.$$

With $\frac{dE}{dn} = V\,\cos\phi$ and $\frac{dE}{dr} = m\,c^2\,\gamma^3\,r/a^2$ this gives:

$$\frac{1}{r}\,\frac{d\varepsilon}{dr} = \frac{d\varepsilon}{dE}\,\frac{dE}{dr} \approx -\frac{V\,m\,c^2\,\gamma^3}{2\,\pi\,N_h\,a^2}\,\left(\frac{d^2\phi}{dE^2}\,\cos\phi - \left(\frac{d\phi}{dE}\right)^2\,\sin\phi\right)\,.$$

Focusing condition ($\sin\phi \approx 0$, factors approx. const):

$$K_x > -\mathrm{const}\,\frac{d^2\phi}{dE^2}\,\cos\phi\,.$$

$\Rightarrow$ **Longitudinal focusing depends on phase curve!**

How does this relate to transition gamma?

Slip factor η (used by synchrotron people;-):

$$\eta = \frac{p}{T}\,\frac{dT}{dp} = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}\,,$$

where γ_t is the "transition gamma". In order to relate γ_t to the parameters ε and ϕ, we derive an expression for γ_t from the definition

$$\gamma_t^2 \equiv \frac{r}{p}\,\frac{dp}{dr}\,.$$

With $B(r) = B_0\,\frac{1+\varepsilon(r)}{\sqrt{1-\frac{r^2}{a^2}}}$ and $p = r\,q\,B(r)$ this gives:

$$\gamma_t^2 = \gamma^2 + r\,\frac{d\varepsilon}{dr}\,.$$

and so in first order:

$$\eta = -r\,\frac{d\varepsilon}{dr}\,.$$

How does this relate to transition gamma?

$$\eta = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}\,,$$

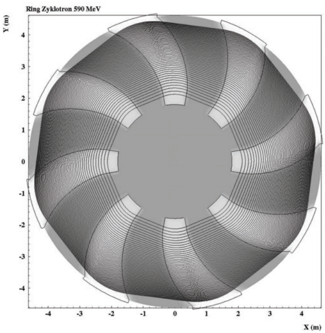
Above transition ($\gamma > \gamma_t$) one has $\eta > 0$, below transition, $\eta < 0$.
Focusing condition is expressed with "slip factor" η:

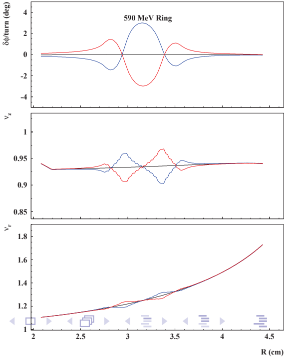
$$K_x > -\frac{\eta}{r^2}$$

Above transition ($\eta > 0$) we have focusing (stability?), below transition we have a threshold.

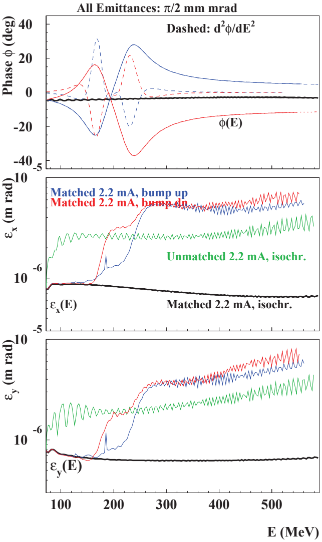
- OPAL [5, 6]: **O**bject oriented **P**arallel **A**ccelerator **L**ibrary developed at PSI (`amas.web.psi.ch`).
- Flavor OPAL -cycl dedicated for the simulation of high intensity cyclotrons.
- Space charge solver: Particle in cell (PIC)-method to compute space charge potential.
- FFT-method for solving electrostatic forces.
- Parallel computing allows to track 10^5 or more particles simultaneously in the cyclotron.
- OPAL uses MAD language with extensions.
- Other flavors for beam transport lines / Linacs available.

1. Create "ideal" ring machine: Geometry similar to ring machine.
2. Adjust perfect or distorted isochronism (see figure).
3. Compute σ-matrix of matched beam distribution for given emittances and beam current [8, 10].
4. Create random Gaussian distribution with 10^5 according to σ-matrix [11].

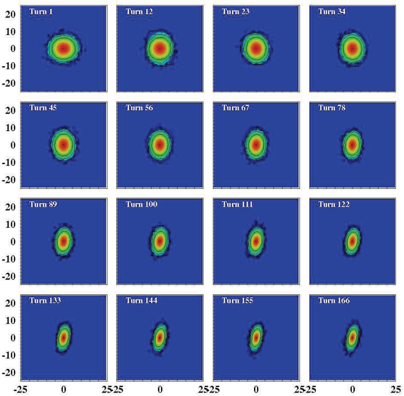


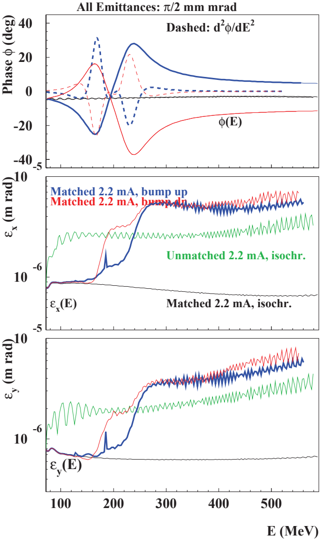


Matched Round Beam in Ideal Cyclotron I

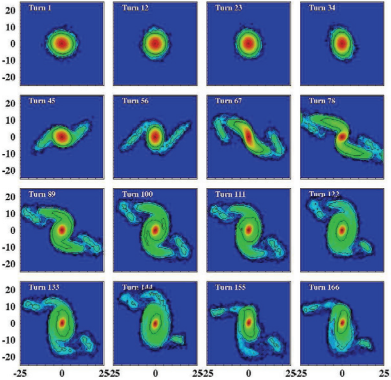


Matched beam, flat phase (black):

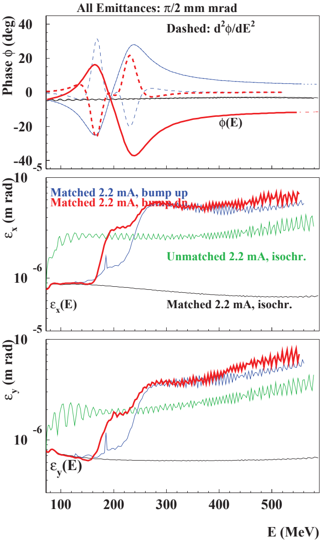




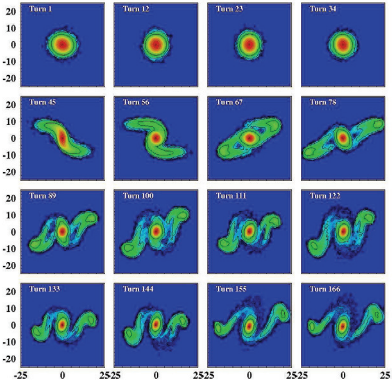
Matched beam, blue phase:



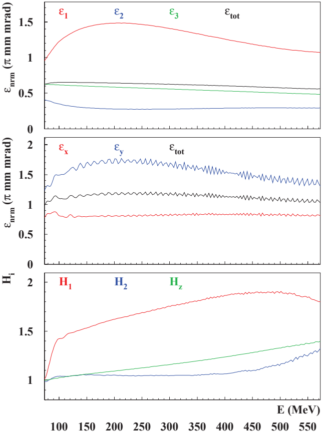
Matched Round Beam in Ideal Cyclotron III



Matched beam, red phase:

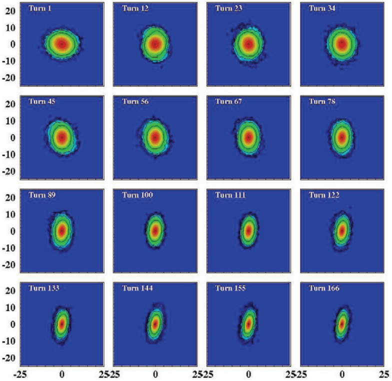


Matched Elliptic Beam in Ideal Cyclotron

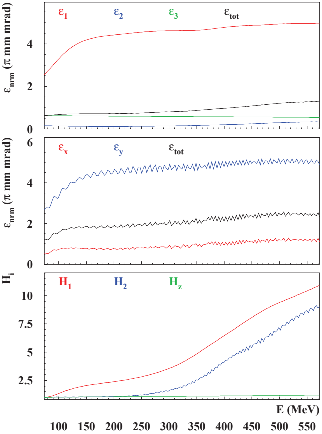


Emittances and Halo-Parameters H_1, H_2, H_3 [7] vs. energy for a 5 mA beam with emittances-ratio of $\frac{2}{3}:1:\frac{3}{2}$. A little bit of spiralling, but bunch stays stable.

Matched elliptic beam, flat phase:

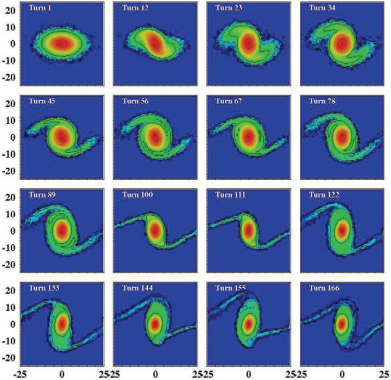


Matched Elliptic Beam in Ideal Cyclotron



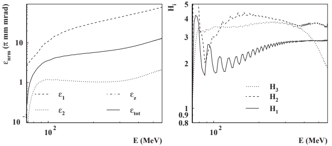
Emittances and Halo-Parameters H_1, H_2, H_3 [7] vs. energy for a $5\,\mathrm{mA}$ beam with emittances-ratio of $\frac{1}{4}$: 1 : 4. Spiralling clearly visible, bunch core stable but of huge halo produced.

Matched elliptic beam, flat phase:

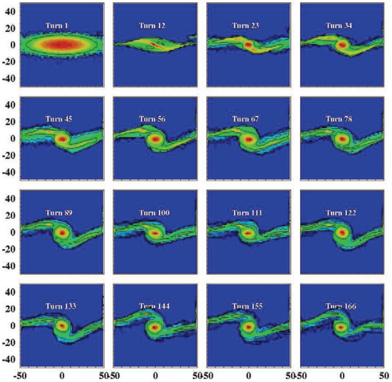


Matched Elliptic Beam in Ideal Cyclotron

Matched elliptic beam, flat phase:



Emittances and Halo-Parameters H_1, H_2, H_3 [7] vs. energy for a $5\,\mathrm{mA}$ beam with emittances-ratio of $\frac{1}{10}$: $\frac{1}{10}$: 12.5. Even though the beam was matched in the linear model, the unequal emittances cause instability.



Result: Matching and focusing are necessary, but not sufficient for bunch compactness. Emittance balancing is also required.

What's "negative mass instability" anyway?

Nielson, Sessler and Symon: "The instability above transition derives from the fact that a force acting on an ion in the direction of motion so as to increase its energy thereby decreases its revolution frequency; in angular coordinate θ the acceleration is in a direction opposite to the direction of applied torque. [...] Since this behavior leads in the wave equation describing the motion of the pertubation to a negative inertia term, we refer to the instability as the negative mass instability."

Pozdeyev, Rodriguez, Marti and York: "Because the beam behavior significantly differed from that predicted by the model of the negative mass instability, the cause of the instability originally was not completely understood. [...] The negative mass instability can only occur if η is positive."

What "negative mass"?

Symplectic decoupling of **F** yields [9]:

$$\tilde{\mathbf{F}} = \mathbf{R}\,\mathbf{F}\,\mathbf{R}^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\beta & 0 & 0 & 0 \\ 0 & 0 & 0 & -\Gamma \\ 0 & 0 & \gamma^2 K_z & 0 \end{pmatrix}$$

with parameters $\beta > 0$ and $\Gamma > 0$. The corresponding Hamiltonian in normal coordinates

$$\tilde{\mathcal{H}}' = \tilde{x}'^2 + \beta\,\tilde{x}^2 - \Gamma\,\tilde{\delta}^2 - \gamma^2 K_z\,\tilde{z}^2$$

contains a negative kinetic energy term. $\Rightarrow$ negative effective mass.

1st: What happens above/below transition?

2nd: What means stability/instability?

Can a Hamiltonian of this form be associated with a stable particle distribution (?):

$$\tilde{\mathcal{H}} = \tilde{x}'^2 + \beta\,\tilde{x}^2 - \Gamma\,\tilde{\delta}^2 - \gamma^2\,K_z\,\tilde{z}^2$$

$\Rightarrow$ Boltzmann equilibrium $\propto \exp\left(-\tilde{\mathcal{H}}/kT\right)$ does not exist!

$\Rightarrow$ No thermodynamic equilibrium: energy can be reduced by emittance transfer into the canonical coordinate with negative effective mass (i.e. momentum spread). This process is linearily suppressed as it is non-symplectic. And it is suppressed, if the emittances (“occupation numbers”) of all degrees of freedom are equal.

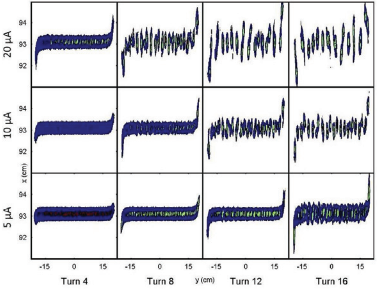
$\Rightarrow$ bunch is “metastable”.

NMI: above or below transition?

Why do Nielson *et al*, Rodriguez *et al* claim instability above transition, where we expect focusing and therefore stability?

- A beam can be (mis-)matched only, if there is focusing in all degrees of freedom.
- Only above transition there is focusing in all degrees of freedom.
- Hence only above transition there can be matched and mismatched beams.
- Thesis: Beam breakup by "negative mass instability" is just a special kind of "mismatching".
- But: When and why does a mismatched beam break up?

NMI = special kind of mismatching?



Beam breakup in small isochronous ring (SIR) [4], caused by "mismatching".

- SIR: Why does the beam "break up" instead of forming a "super-ellipsoid"?
- Longitudinal focusing has no unique "zero" position; only relative to the bunch center.
- Very long bunches have no sharp or well-defined "bunch center".
- SIR: a strongly mismatched beam breaks up into a number of matched self-focusing bunches.
- ⇒ The "instability" might be re-interpreted as "mismatching".

Effect of Longitudinal Defocusing

1. If the longitudinal focusing frequency is imaginary, the beam expands longitudinally.
2. The horizontal-longitudinal coupling increases also horizontal beam size.
3. The beam expansion reduces space charge forces.
4. The reduced space charge forces reduce focusing.
5. Without flat-top cavity:
6. $\Rightarrow$ filamentation $\Rightarrow$ **irreversible** increase of emittance.
7. $\Rightarrow$ increased extraction losses.

Effect of Longitudinal Focusing

1. If space charge is sufficiently strong and the field isochronous, the beam is longitudinally focused.
2. A focused beam must be matched and the emittances balanced in order to stay compact.
3. In a mismatched or unbalanced beam the spiralling caused by (non-linear) space charge produces a halo.
4. In case of strong mismatching or imbalance the emittance increase is large enough to destroy focusing.
5. Even in case of a matched and balanced beam, a poor isochronism or a passage of the transition can destroy the focusing.

Conclusions

First conclusion: "Simple" linear matching model works - with some additional considerations. But: Iteration required for accurate solution [8].

If high power cyclotrons ("dream machines") are supposed to take advantage of longitudinal focusing by space charge, ...

- ...the injected beam should be matched.
- ...the emittances should be balanced.
- ...the phase curve must be sufficiently flat over all turns.
- ...a high beam brightness is required (PSI-Ring: $\varepsilon \leq 1.5\,\mu\,m\,rad$ at $2.2\,\mathrm{mA}$).
- ...the focusing frequency ν_z should be as high as possible.
- ...the cyclotron optics should be simulated before the finalization of design.

Thank you for your attention.

Thanks to R. Dölling, M. Humbel and Hui Zhang for the Time Structure Measurements.
Thanks to A. Adelmann for helping with OPAL .

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