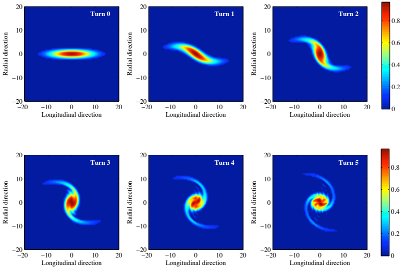
Vlasov equation approach to space charge effects in isochronous cyclotrons



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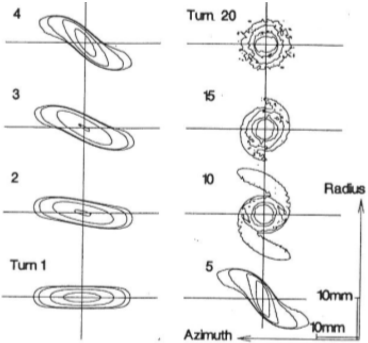
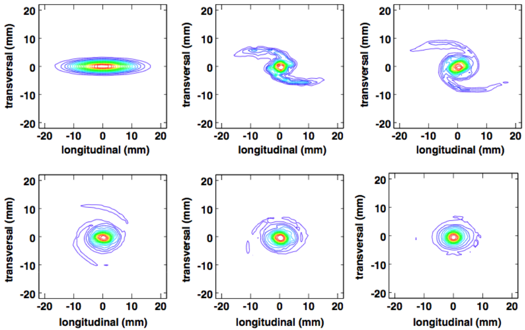
Motivation: Spiraling in isochronous machines

PSI Injector II

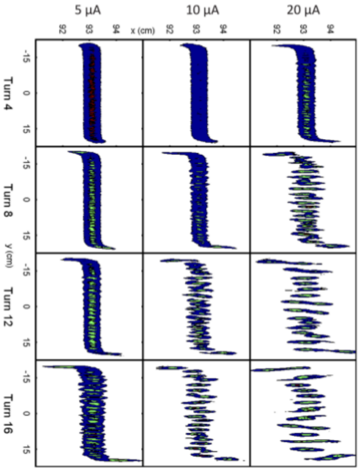


PICN (Adam)

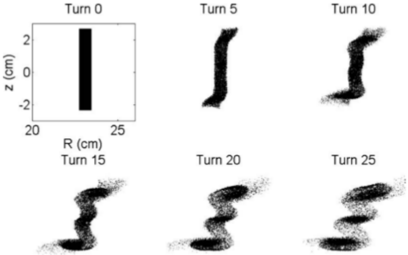


OPAL-CYCL (Adelmann *et al.*)

Motivation: Breakup in isochronous machines



Small Isochronous Ring
(Pozdeyev *et al.*)



CYCIAE 100
(Bi *et al.*)

Motivation

- Beam spiraling and beam breakup
  - Observed in PIC simulations
  - Indirect experimental observations
  - Single-particle explanation
- PIC simulations
  - Combine Newton's equations with Maxwell's equations
  - Strengths: Conceptually simple, reliable tool, moves the theoretical effort from physics to computer science
  - Weaknesses: Does not provide intuitive understanding, computationally costly
- Single particle picture: how to extend it for the nonlinear dynamics?
- **Can we use continuum kinetic theory for an intuitive and analytic explanation of these effects?**

Outline

- **Fluid theory of beam vortex motion**
  - Fluid model for highly intense proton beams
  - Multiple time scale analysis and averaging procedure
- **Beam stability**
  - Comparison with PIC simulations
  - Isomorphism with Euler equations for a fluid
  - Beam stability theorems

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Fluid theory of beam vortex motion

Our fluid theory is based on two principles:

1. The principle of **maximal** geometric simplification:
   - **Homogeneous magnetic field**: $\mathbf{B} = B\mathbf{e}_z$
   - **Non-relativistic coasting beam**
   - **Two-dimensional problem**: $\partial/\partial z \equiv 0$ for all quantities
2. The principle of **minimal** beam physics simplification

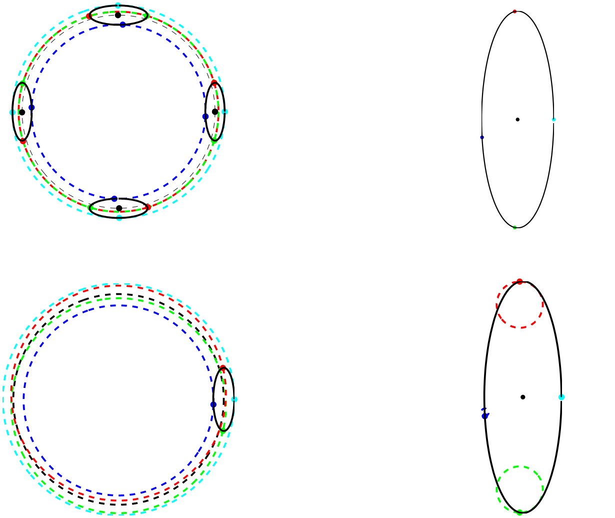
   We will derive all the results from **kinetic theory** and the **Vlasov** equation, using only **one** assumption:

   The amplitude of the mismatch oscillations is small compared to the size of the proton beam (i.e. departure from laminar flow is small)

Lab frame **Moving frame**

Laminar

Departure from laminar regime



How small is small?

- Let a be the characteristic size of the beam; n the beam density; v_T the beam thermal velocity; ω_c the cyclotron frequency
- Relative size of space charge force and magnetic force

$$\left|\frac{\mathbf{E}}{\mathbf{v} \times \mathbf{B}}\right| \sim \frac{e^2 a n}{m \epsilon_0 v_T \omega_c} \sim \frac{e^2 n}{m \epsilon_0 \omega_c^2} \equiv \frac{\omega_p^2}{\omega_c^2}$$

  where ω_p is the plasma frequency
- Define

$$\delta^2 = \frac{\omega_p^2}{\omega_c^2} = \frac{mn}{\epsilon_0 B^2}$$

- All cyclotrons and rings satisfy $\delta^2 \leq 1$, and **most satisfy $\boldsymbol{\delta^2 \ll 1}$**
- We consider the regime in which the **mismatch oscillation amplitude is of order $\boldsymbol{\delta}$a**

Deriving fluid equations for the beam

- Evolution of the beam **distribution function** $f(\mathbf{x}, \mathbf{v}, t)$ in phase space given by the Vlasov equation:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{e}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f = 0$$

- Taking the integrals $\int d\mathbf{v}$ and $\int \mathbf{v} d\mathbf{v}$ of this equation, and defining $n \equiv \int f d\mathbf{v}$, $n\mathbf{V} = \int \mathbf{v} f d\mathbf{v}$, and $\mathbf{P} = m \int (\mathbf{v} - \mathbf{V})(\mathbf{v} - \mathbf{V}) f d\mathbf{v}$, we can obtain **fluid-like equations**:

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{V}) = 0 \qquad \text{Continuity}$$

$$mn \left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} \right) = en (\mathbf{E} + \mathbf{V} \times \mathbf{B}) - \nabla \cdot \mathbf{P} \qquad \text{Momentum}$$

- Well-known **closure problem**: too many unknowns and not enough equations
  $\Rightarrow$ In general, need **kinetic codes** to solve the Vlasov equation

Pressure tensor

- When $\delta^2 \ll 1$ and the mismatch oscillation amplitude is of order δa, we can derive **closed fluid equations**
- That's because to lowest order in δ the tensor $\mathbf{P}$ has the form

$$\mathbf{P} = p_\perp \mathbf{I} + (p_\parallel - p_\perp)\mathbf{e}_z\mathbf{e}_z = \begin{pmatrix} p_\perp & 0 & 0 \\ 0 & p_\perp & 0 \\ 0 & 0 & p_\parallel \end{pmatrix}$$

- Proof: • Compare all the terms to the magnetic term in the Vlasov equation, with $|\mathbf{v}| \sim \delta v_T$:

$$\frac{\mathbf{E}}{\mathbf{v}\times\mathbf{B}} \sim \frac{aen}{\delta a \omega_c \epsilon_0 B} \sim \frac{\omega_p^2}{\delta\omega_c^2} \sim \delta$$

$$\frac{\mathbf{v}\cdot\nabla f}{e/m\mathbf{v}\times\mathbf{B}\cdot\nabla_\mathbf{v} f} \sim \frac{\delta\omega_c}{\omega_c} \sim \delta \qquad \frac{\partial f/\partial t}{e/m\mathbf{v}\times\mathbf{B}\cdot\nabla_\mathbf{v} f} \sim \frac{v}{a\omega_c} \sim \delta$$

  • To lowest order in δ, the Vlasov equation therefore is

$$\mathbf{v}\times\mathbf{B}\cdot\nabla_\mathbf{v} f = 0 \qquad \Rightarrow \partial f/\partial\varphi = 0 \qquad (\varphi : \text{gyrophase})$$

Fluid equations

$$\nabla \cdot \mathbf{P} = \nabla p_\perp + (p_\parallel - p_\perp)\mathbf{e}_z \cdot \nabla \mathbf{e}_z + \mathbf{e}_z \cdot \nabla (p_\parallel - p_\perp)\mathbf{e}_z + (p_\parallel - p_\perp)\nabla \cdot \mathbf{e}_z\mathbf{e}_z$$
$$= \nabla p_\perp + \mathbf{e}_z \cdot \nabla (p_\parallel - p_\perp)\mathbf{e}_z$$

- In the radial-longitudinal plane, **only contribution is $\nabla p_\perp$**
- **Fluid equations in the frame moving with the beam for the 2D motion in the radial-longitudinal plane:**

$$\frac{dn}{dt} + n\nabla \cdot \mathbf{v} = 0 \qquad \frac{d\mathbf{v}}{dt} + \mathbf{v} \times \mathbf{e}_z = -\delta^2 \left(\nabla\phi + \frac{\alpha^2}{n}\nabla p_\perp \right)$$
$$\nabla^2\phi = -n \qquad \alpha^2 \equiv T_{max}/ma^2\omega_p^2$$

- IMPORTANT: The pressure term comes in as an **exact gradient**; as we prove later, this means that we **do not need an equation for the evolution of $p_\perp$**

Multiple time scale analysis

- When $\delta^2 \ll 1$, the space charge time scale is **much longer** than the betatron time scale
- The particle motion is **quasi-periodic**
- This can be used to reduce the complexity of numerical simulations (e.g. PICS)
- We use it to derive **simple fluid equations valid on the space charge time scale**, using a multiple time scale analysis

- Step 1: Each quantity Q is assumed to vary according to the different time scales as follows:

$$Q(\mathbf{r}, t) = Q(\mathbf{r}, t_0, t_2, t_4, \ldots) = Q(\mathbf{r}, t, \delta^2 t, \delta^4 t, \ldots)$$

  t_0 is the betatron time scale; $t_2 \sim \delta^2 t_0$ is the space charge time scale

$$\frac{\partial Q}{\partial t} = \frac{\partial Q}{\partial t_0} + \delta^2 \frac{\partial Q}{\partial t_2} + \ldots$$

- Step 2: Define averaging operation over the betatron time scale

$$\langle Q \rangle = \frac{1}{2\pi} \int_0^{2\pi} Q(\mathbf{r}, t_0, t_2, \ldots) dt_0$$

  Separate Q into the sum of a rapidly oscillating part $\tilde{Q}$, and a slow monotonic evolution $\bar{Q}$:

$$Q(\mathbf{r}, t_0, t_2, \ldots) = \tilde{Q}(\mathbf{r}, t_0, t_2, \ldots) + \bar{Q}(\mathbf{r}, t_2, \ldots)$$

  where by definition $\langle \tilde{Q} \rangle = 0$

- Step 3: Expand all quantities in δ. The expansion corresponding to our assumptions is

$$
\begin{aligned}
n &= \bar{n_0} + \delta\left(\tilde{n_1} + \bar{n_1}\right) + \delta^2\left(\tilde{n_2} + \bar{n_2}\right) + O(\delta^3) \\
p_\perp &= \bar{p_0} + \delta\left(\tilde{p_1} + \bar{p_1}\right) + \delta^2\left(\tilde{p_2} + \bar{p_2}\right) + O(\delta^3) \\
\phi &= \bar{\phi_0} + \delta\left(\tilde{\phi_1} + \bar{\phi_1}\right) + \delta^2\left(\tilde{\phi_2} + \bar{\phi_2}\right) + O(\delta^3) \\
\mathbf{v} &= \delta\tilde{\mathbf{v}}_1 + \delta^2\left(\tilde{\mathbf{v}}_2 + \bar{\mathbf{v}}_2\right) + O(\delta^3)
\end{aligned}
$$

- Step 4: Plug this expansion into the fluid equations and solve order by order in δ. For the density, we have

$$
\begin{aligned}
&\frac{\partial\tilde{n_1}}{\partial t_0} + \nabla\cdot\left(\bar{n_0}\tilde{\mathbf{v}}_1\right) = 0 \quad O(\delta) \\
&\frac{\partial\tilde{n_2}}{\partial t_0} + \frac{\partial\bar{n_0}}{\partial t_2} + \nabla\cdot\left[\left(\tilde{n_1} + \bar{n_1}\right)\tilde{\mathbf{v}}_1 + \bar{n_0}\left(\tilde{\mathbf{v}}_2 + \bar{\mathbf{v}}_2\right)\right] = 0 \quad O(\delta^2) \\
&\frac{\partial\bar{n_0}}{\partial t_2} + \nabla\cdot\left(< \tilde{n_1}\tilde{\mathbf{v}}_1 > + \bar{n_0}\bar{\mathbf{v}}_2\right) = 0
\end{aligned}
$$

- $\tilde{n}_1$ and $\tilde{\mathbf{v}}_1$ are given by the lowest order betatron motion: easy to compute [1]
- Taking the momentum equation to $O(\delta^2)$ and averaging it on the fast time scale, we find:

$$\bar{\mathbf{v}}_2 = \langle \tilde{\mathbf{v}}_1 \cdot \nabla \tilde{\mathbf{v}}_1 \rangle \times \mathbf{e}_z + \nabla \bar{\phi}_0 \times \mathbf{e}_z + \frac{\alpha^2}{\bar{n}_0} \nabla \bar{p}_0 \times \mathbf{e}_z$$

- Combining all the results, we find after some algebra[1]

$$\frac{\partial \bar{n}_0}{\partial t_2} + \nabla \cdot \left(\bar{n}_0 \nabla \bar{\phi}_0 \times \mathbf{e}_z \right) + \alpha^2 \nabla \cdot \left[\nabla \times \left(\bar{p}_0 \mathbf{e}_z \right) \right] = 0$$

$$\Leftrightarrow \frac{\partial \bar{n}_0}{\partial t_2} + \nabla \bar{\phi}_0 \times \mathbf{e}_z \cdot \nabla \bar{n}_0 = 0$$

- **In our ordering, temperature effects do not play any effect on the slow time scale** (at least to lowest order)

[1] A.J. Cerfon, J.P. Freidberg, F.I. Parra, and T.A. Antaya, PRSTAB **16**, 024202 (2013)

Beam dynamics due to space charge effects

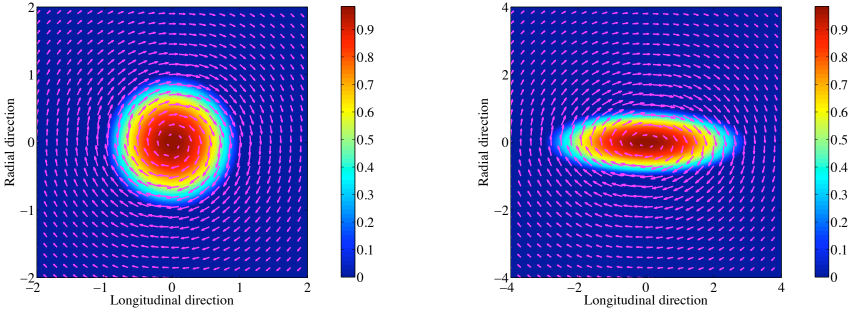
- Final result:

$$\frac{\partial \bar{n}_0}{\partial t_2} + \nabla \bar{\phi}_0 \times \mathbf{e}_z \cdot \nabla \bar{n}_0 = 0$$
$$\nabla^2 \bar{\phi}_0 = -\bar{n}_0$$

- Describes the advection of the density profile in the velocity field $\mathbf{E} \times \mathbf{B}/B^2$
- Agrees with single-particle picture, and extends it to the nonlinear regime
- Our result is a first-principle derivation of Gordon's[2] intuition
- **δ^2 only appears through t_2**. Bunches with different densities **have identical behavior**. Only difference: growth rates and frequency scale **linearly with n**

[2]M.M. Gordon, in *Proceedings of the 5th International Cyclotron Conference*, Oxford 1969, pp. 305-317

ExB advection



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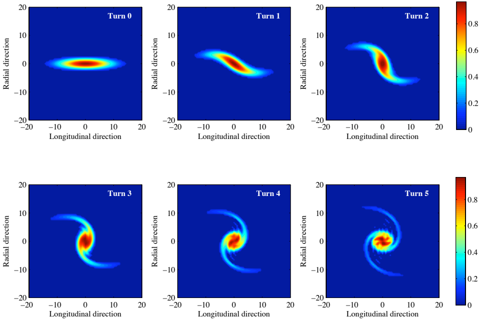
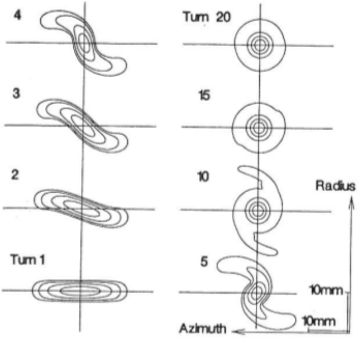
IS OUR MODEL RELEVANT?

Our simulation



PICS simulation



- Good agreement, even at $\delta^2 = 0.8$
- Geometrical effects play a very limited role

Isomorphism with 2D Euler equations

Beam vortex dynamics

$$\frac{\partial n}{\partial t} + \nabla\phi \times \mathbf{e}_z \cdot \nabla n = 0$$
$$\nabla^2\phi = -n$$

n: bunch density; ϕ: electrostatic potential

2D incompressible Euler

$$\frac{\partial \omega}{\partial t} + \nabla\psi \times \mathbf{e}_z \cdot \nabla \omega = 0$$
$$\nabla^2\psi = -\omega$$

ω: z-directed vorticity; ψ: stream function for the flow

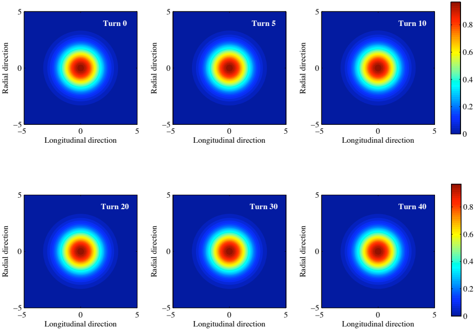
- Isomorphism recognized a long time ago in a slightly different context[3]
- We proved that the isomorphism holds even for finite temperature beams
- We can use decades old fluid dynamics results to determine/understand the stability of bunch distributions

[3] C.F. Driscoll and K.S. Fine, *Phys.Fluids B* **2** 1359 (1990)

Stability of round beams

- Radial density distributions automatically satisfy the equations
- Well-known results from fluid theory of radially symmetric vortex patches:
  - If $n(r)$ is **monotonically decreasing**, the bunch is **nonlinearly stable** to nonsymmetric density perturbations
  - Hollow density profiles can be unstable to these perturbations

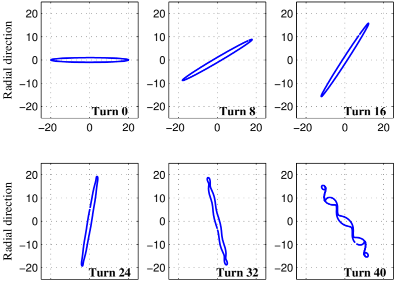
Gaussian $n(r)$, $\delta^2 = 0.8$



ELLIPTIC BUNCHES WITH UNIFORM DENSITY

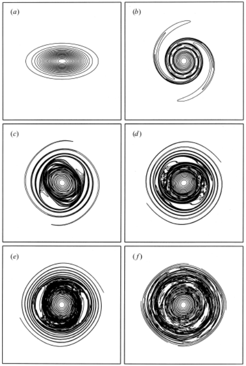
- Classical case in fluid dynamics: **uniform density profile**
  Call a the semi-major axis and b the semi-minor axis
- If $a/b < 3$, bunch is linearly and nonlinearly **stable** to edge perturbations
  If $a/b > 3$, bunch is linearly and nonlinearly **unstable** to edge perturbations
- Instability is a potential mechanism for **beam breakup**

Uniform n, $\delta^2 = 0.2$, $a/b = 20$



Elliptic bunches with smooth density profile

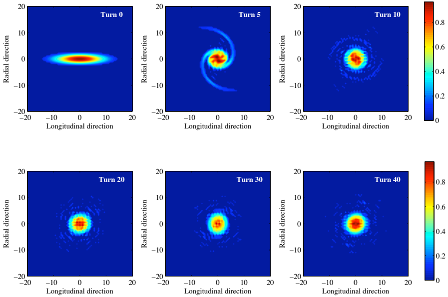
- More complicated case. Answer depends on the smoothness of the profile
- For reasonably smooth profile, **"axisymmetrization principle"**[4] even for $a/b < 3$



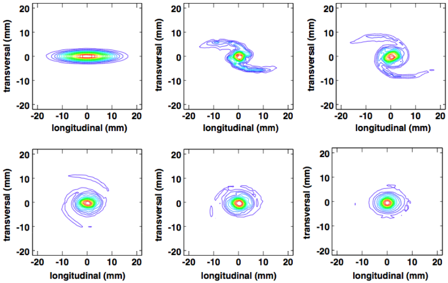
[4] M.V. Melander, J.C. McWilliams and N.J. Zabusky, J. Fluid Mech. 178 (1987) 137

Beam spiraling a.k.a axisymmetrization

Our simulation



OPAL simulation



- In PSI Injector II, formation of a round stable core after 40 turns (good)
- In machines with lower δ^2, core and halo take longer to form Potentially bad situation if low density halo forms with high energy

Summary

- Space-charge forces are small relative to magnetic forces by the ratio $\delta^2 = \omega_p^2/\omega_c^2 \leq 1$
- The scale separation between betatron and space charge time scales can be advantageously used to reduce the complexity of kinetic calculations and reduce computational time
- When the mismatch amplitude is small by δ compared to the typical size of the beam, a fluid model of the beam can be rigourously derived from the Vlasov equation
- In the fluid model, beam spiraling is a consequence of the advection of the beam in the $\mathbf{E} \times \mathbf{B}$ velocity field
- Bunches behave like isolated vortex distributions in the 2D incompressible Euler equations
- This physical picture is in quantitative agreement with PIC simulations

Ongoing and future work

- Include accelerating gaps. Design accelerating voltage shape/phase to counter spiraling?
- Allow arbitrary departure from laminar regime
  - Requires numerically solving a kinetic equation
  - Idea: use scale separation between betatron and space charge time scales to simplify Vlasov equation
  - Write a reduced continuum kinetic code
- Relativistic regime (Adelmann)
- Consider realistic magnetic field configurations and 3D effects
- **Develop collaboration with experimentalists and PIC developers**
  Suggestions you might have? Lets discuss them!