VLASOV EQUATION APPROACH TO SPACE CHARGE EFFECTS IN ISOCHRONOUS MACHINES

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Abstract

Starting from the collisionless Vlasov equation, we derive two simple coupled two-dimensional partial differential equations describing the radial-longitudinal beam vortex motion associated with space charge effects in isochronous cyclotrons. These equations show that the vortex motion can be intuitively understood as the nonlinear advection of the beam by the $\mathbf{E} \times \mathbf{B}$ velocity field, where $\mathbf{E}$ is the electric field due to the space charge and $\mathbf{B}$ is the applied magnetic field. The partial differential equations are also formally identical to the two-dimensional Euler equations for a fluid of uniform density. From this analogy, we explain why elongated beams develop spiral halos and a stable round core while round beams are always stable. Solving the coupled equations numerically, we find good agreement between our model and Particle-In-Cell simulations.

INTRODUCTION

Modern applications of cyclotrons in fields such as materials science, nuclear medicine and national security require high quality beams at high intensities. In order to design cyclotrons that are capable of producing such beams reliably, one needs to be able to understand and predict the effects of space charge forces on beam evolution. Because of the complexity of beam dynamics and of the magnetic field geometry in modern machines, the theoretical work is now largely numerical.

Recently, very sophisticated numerical tools based on the Particle-In-Cell (PIC) method have been developed to design modern cyclotrons and analyze experiments [1, 2, 3]. Codes based on the PIC approach are very intuitive and can be conveniently parallelized for large-scale computations. These are critical points given the complexity of the problem at hand. The evolution of charged particle beams in high intensity cyclotrons is indeed described by the seven-dimensional Vlasov-Poisson system in complicated geometries [1]. However, there are also drawbacks with direct PIC simulations of the exact Vlasov-Poisson system. First, precisely because of the conceptual simplicity of the PIC formulation, it often does not provide insights on the basic phenomena involved in the dynamics until one runs the simulations and extracts information from the numerical results. More importantly, the conventional PIC method is subject to difficulties associated with statistical noise, which require the simulation of a very large number of particles to bring the statistical uncertainty to an appropriately low level. Accurate fully self-consistent PIC simulations of beam dynamics in modern cyclotrons are thus very computationally intensive [1]. While these high performance numerical tools are very valuable for in-depth calculations, the computational times – measured in hours – are impractical for fast scoping studies of novel designs and for the fast interpretation of results during experimental runs.

In a wide class of cyclotrons, space charge forces are important because of their cumulative effect after many turns, but they only slightly disturb the single-particle dynamics on a given revolution. For these machines, the quasi-periodicity of the single-particle motion can form the basis for an averaging procedure that reduces time resolution requirements and the dimensionality of the Vlasov-Poisson problem. Adam relied on this fact to develop the successful two-dimensional PIC code PICS based on the “Sphere Model” [4]. A very similar averaging procedure known under the name gyrokinetics (e.g. [5]) is also succesfully used in plasma physics for accurate and tractable simulations of fusion and astrophysical plasmas [6, 7, 8, 9].

This paper belongs to the early stage of an ongoing project to derive averaged equations for the Vlasov-Poisson system in modern cyclotrons, and develop a continuum kinetic code to solve these equations and study space-charge effects. While we will treat the general case in future work, we focus here on the particular regime in which the beam is almost laminar. We show that in this regime the averaging procedure can be used to obtain a reduced set of fluid equations describing the non-relativistic radial-longitudinal beam dynamics due to space charges. These fluid equations have the remarkable property to be isomorphic to the two-dimensional Euler equations for a fluid of uniform density. This analogy is very powerful to understand the stability of given beam density distributions, and explain beam spiraling [1, 4] and breakup phenomena [10] observed in isochronous cyclotrons.

REDUCED FLUID MODEL FOR THE BEAM

Starting Model

In this work, we make a number of simplifying assumptions that we will relax in future work. These assumptions allow us to make substantial analytic progress, and as we will show below, they lead to a model than contains all the key physics describing beam spiraling and beam breakup.

In order to compare our results with previous numerical results [1, 4], we focus on the case of a coasting beam. We restrict our study to the two-dimensional radial-longitudinal plane, and consider a non-relativistic beam. With these three restrictions in mind, we can consider the

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simple case in which the cyclotron magnetic field is homogeneous. We write $\mathbf{B} = B\mathbf{e}_z$, where B is a constant and $\mathbf{e}_z$ is the unit vector along the vertical direction. For the simplicity of the notation, we apply our analysis to a proton beam, but all the results can be easily modified for particles with a different charge and a different mass. Finally, as is common [1] and justified given the low density of cyclotron beams, we neglect binary collisions between protons in the beams. The evolution of the proton distribution function is therefore given by the Vlasov equation, which is the starting point of our analysis. The first step is to derive fluid equations for the beam from the Vlasov equation.

We carry out our analysis in a frame rotating with the beam. In this frame, the electrostatic approximation holds so that the electric field due to the space charges can be written as $\mathbf{E} = -\nabla\phi$. The magnetic self fields can be neglected since we are in the non-relativistic limit. Taking the first two velocity moments of the Vlasov equation [11], we then get the following two fluid-like equations for the evolution of the beam, in non-dimensionalized form [12]

$$\frac{dn}{dt} + n\nabla\cdot\mathbf{v} = 0 \tag{1}$$

$$\frac{d\mathbf{v}}{dt} + \mathbf{v}\times\mathbf{e}_z = -\delta^2\left(\nabla\phi + \frac{\alpha^2}{n}\nabla\cdot\mathbf{P}\right) \tag{2}$$

where ϕ is computed from the non-dimensionalized Poisson equation

$$\nabla^2\phi = -n \tag{3}$$

In Eqs. 1 and 2, $n(\mathbf{r},t)$ is the normalized beam density, $\mathbf{v}(\mathbf{r},t)$ its normalized velocity, and $\mathbf{P}(\mathbf{r},t)$ its normalized pressure tensor. $d/dt \equiv \partial/\partial t + \mathbf{v}\cdot\nabla$ and $\delta^2 = \omega_p^2/\omega_c^2$ where ω_p is the plasma frequency at the peak beam density, $\omega_p^2 = N_0e^2/m\epsilon_0$. α is defined by $\alpha^2 = T_0/ma^2\omega_p^2 = \lambda_D^2/a^2$ where λ_D is the Debye length. N_0 is the peak density of the beam, T_0 its peak temperature, and a its mean radius. δ^2 represents the strength of space-charge effects relative to the confining magnetic force, while α^2 represents the strength of temperature effects relative to space-charge effects.

Many cyclotrons operate in the regime $\delta^2 \ll 1$. In this regime, the motion of the charged particles consists of the periodic betatron motion plus a small perturbation due to space charge forces. We will use this fact and perturbation theory to average Equations 1, 2 and 3 over the quasi-periodic particle orbits and derive simplified fluid equations that capture the slow, betatron-averaged evolution of the beam under the effect of space charges. Before doing so, we need a closure for the fluid-like equations. Here too the smallness of δ can be used, in order to derive a convenient approximation for the pressure term in Eq. 2. This is what we do in the next section.

Closing the Fluid Equations

The system formed by Equations 1, 2 and 3 is not closed since it does not include a prescription on how to evolve the pressure tensor $\mathbf{P}$. In general, $\mathbf{P}$ can only be computed by directly solving the Vlasov equation. This is the reason why complex seven dimensional numerical solvers are usually required. In order to make further analytical progress, we instead assume that the ratio of the amplitude of the betatron oscillations over the mean beam radius is of order the small parameter δ. This is equivalent to saying that the beam is mismatched but that the departure from the laminar regime is small. When this is the case, it can be shown by ordering the different terms in the Vlasov equation [12] that to lowest order in δ the pressure tensor can be written as

$$\mathbf{P} = p_\perp\mathbf{I} + (p_z - p_\perp)\mathbf{e}_z\mathbf{e}_z \tag{4}$$

where $\mathbf{I}$ is the unit tensor, and $\perp$ denotes the plane perpendicular to $\mathbf{e}_z$, i.e. the radial-longitudinal plane. With this particular form of the pressure tensor, the only contribution of $\nabla\cdot\mathbf{P}$ that is not in the $\mathbf{e}_z$ direction is $\nabla p_\perp$. Our fluid model now takes the following form

$$\frac{dn}{dt} + n\nabla\cdot\mathbf{v} = 0 \tag{5}$$

$$\frac{d\mathbf{v}}{dt} + \mathbf{v}\times\mathbf{e}_z = -\delta^2\left(\nabla\phi + \frac{\alpha^2}{n}\nabla p_\perp\right) \tag{6}$$

$$\nabla^2\phi = -n \tag{7}$$

In principle the system of Equations 5 – 7 is still not closed because it does not contain an equation for the evolution of $p_\perp$. However, one of the remarkable results of the averaging procedure in the next section is that such an equation is in fact unnecessary. It turns out that when the pressure term can be written as a gradient, it vanishes exactly from the equations for the evolution of the beam on the space-charge time scale.

Averaging over the Betatron Period

The averaging procedure is based on the fact that the betatron time scale is shorter than the space-charge time scale by the small factor δ^2, and is carried out as follows. First, one formally separates the time dependence of any quantity Q according to

$$Q(\mathbf{r},t) = Q(\mathbf{r},t_0,t_2,\ldots) = Q(\mathbf{r},t,\delta^2 t,\ldots) \tag{8}$$

where t_0 corresponds to the betatron time scale and t_2 to the slower space-charge time scale. With this formal scale separation we have

$$\frac{\partial Q}{\partial t} = \frac{\partial Q}{\partial t_0} + \delta^2\frac{\partial Q}{\partial t_2} + O(\delta^4) \tag{9}$$

It is then convenient to define the averaging operation over the betatron time scale as follows

$$\langle Q\rangle \equiv \frac{1}{2\pi}\int_0^{2\pi} Q\,dt_0$$

and separate any quantity Q into the sum of a rapidly oscillating part $\tilde{Q}$ due to the betatron oscillations, and a slowly evolving part $\bar{Q}$ due to space-charge and thermal effects:

$$Q(\mathbf{r},t_0,t_2,\ldots) = \tilde{Q}(\mathbf{r},t_0,t_2,\ldots) + \bar{Q}(\mathbf{r},t_2,\ldots) \tag{10}$$

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By construction $\langle\tilde{Q}\rangle = 0$ and $\bar{Q}$ does not depend on the fast time scale t_0.

The next step in the analysis is to expand n, $\mathbf{v}$, $p_\perp$ and ϕ in powers of δ. The ordering that is consistent with the assumption that the departure from the laminar regime is small is

$$n = \bar{n}_0 + \delta\left(\tilde{n}_1 + \bar{n}_1\right) + \delta^2\left(\tilde{n}_2 + \bar{n}_2\right) + O(\delta^3) \quad (11)$$

$$p_\perp = \bar{p}_{\perp 0} + O(\delta) \quad (12)$$

$$\phi = \bar{\phi}_0 + O(\delta) \quad (13)$$

$$\mathbf{v} = \delta\tilde{\mathbf{v}}_1 + \delta^2\left(\tilde{\mathbf{v}}_2 + \bar{\mathbf{v}}_2\right) + O(\delta^3) \quad (14)$$

We then introduce these expansions into Equations 5 – 7. We start with Poisson's equation, which we only need to lowest order in δ:

$$\nabla^2\bar{\phi}_0 = -\bar{n}_0 \quad (15)$$

We continue with Eq. 5. The first nontrivial contribution to this equation arises in first order in δ:

$$\frac{\partial\tilde{n}_1}{\partial t_0} + \nabla\cdot\left(\bar{n}_0\tilde{\mathbf{v}}_1\right) = 0 \quad (16)$$

Equation 16 describes the evolution of the beam density (to lowest order) on the fast time scale, under the effect of the betatron oscillations. To second order in δ, Eq. 5 is

$$\frac{\partial\tilde{n}_2}{\partial t_0} + \frac{\partial\bar{n}_0}{\partial t_2} + \nabla\cdot\left[\left(\tilde{n}_1 + \bar{n}_1\right)\tilde{\mathbf{v}}_1 + \bar{n}_0\left(\tilde{\mathbf{v}}_2 + \bar{\mathbf{v}}_2\right)\right] = 0 \quad (17)$$

The evolution of the beam density on the space charge time scale is obtained by averaging Eq. 17 over the betatron time scale. We find

$$\frac{\partial\bar{n}_0}{\partial t_2} + \nabla\cdot\left(\langle\tilde{n}_1\tilde{\mathbf{v}}_1\rangle + \bar{n}_0\bar{\mathbf{v}}_2\right) = 0 \quad (18)$$

Equation 18 is precisely the equation we were looking for: it determines the evolution of the beam density due to space charge forces and the averaged effect of the betatron motion. The evolution of $\tilde{n}_1$ appearing in Eq. 18 is given by Eq. 16. However, we still need equations for $\tilde{\mathbf{v}}_1$ and $\bar{\mathbf{v}}_2$, which we get by also expanding the momentum equation order by order in δ. The first nontrivial contribution comes in first order in δ:

$$\frac{\partial\tilde{\mathbf{v}}_1}{\partial t_0} + \tilde{\mathbf{v}}_1 \times \mathbf{e}_z = \mathbf{0} \quad (19)$$

This is the desired equation for $\tilde{\mathbf{v}}_1$, simply describing the betatron motion. In order to obtain an equation for $\bar{\mathbf{v}}_2$, we take the momentum equation to second order in δ and average it over t_0. We find

$$\bar{\mathbf{v}}_2 = \langle\tilde{\mathbf{v}}_1\cdot\nabla\tilde{\mathbf{v}}_1\rangle \times \mathbf{e}_z + \nabla\bar{\phi}_0 \times \mathbf{e}_z + \frac{\alpha^2}{\bar{n}_0}\nabla\bar{p}_0 \times \mathbf{e}_z \quad (20)$$

All the relevant equations have now been derived. From Eqs.16, 19 and 20 we can derive expressions for $\tilde{n}_1$, $\tilde{\mathbf{v}}_1$ and $\bar{\mathbf{v}}_2$ in terms of zeroth order quantities and initial conditions, which we can then insert in Eq. 18 to obtain the desired equation for the evolution of the beam density on the space-charge time scale. The details of the calculation are slightly tedious and can be found in [12]. The final result is the following coupled system obtained from Eqs. 15 and 18:

$$\frac{\partial\bar{n}_0}{\partial t_2} + \nabla\bar{\phi}_0 \times \mathbf{e}_z \cdot \nabla\bar{n}_0 = 0 \quad (21)$$

$$\nabla^2\bar{\phi}_0 = -\bar{n}_0 \quad (22)$$

Note that the pressure term is absent from these equations.

Isomorphism with Two-dimensional Incompressible Euler Equations

Equations 21 and 22 have a simple interpretation: they describe the advection of the beam in the $\mathbf{E}\times\mathbf{B}$ velocity field, where $\mathbf{E} = -\nabla\phi$ is the electric field due to the space charges [12]. They are remarkable in that they are formally identical to the two-dimensional Euler equations for a fluid with uniform density. In the Euler case, the vorticity plays the role of the density n, and the stream function for the flow plays the role of the electrostatic potential ϕ. Because n is always positive, the analogy is restricted to two-dimensional fluid vortex dynamics with positive vorticity. The analogy gives a robust theoretical framework to the concept of beam vortex motion first pointed out by Gordon [13]. A similar analogy was identified a long time ago regarding the two-dimensional dynamics of electrons magnetically confined in a plasma column [14] (and references therein). This is not surprising since Eqs. 5 – 7 are identical to those describing the evolution of an isotropic electron beam in the plane perpendicular to the homogeneous confining magnetic field. Our derivation generalizes these results by showing that the analogy even holds for finite temperarure beams. In other words, with our ordering betatron oscillations have no effect on the evolution of the beam density on the slow time scale, as they average out.

RADIAL - LONGITUDINAL BEAM DYNAMICS

The isomorphism between Eqs. 21 and 22 and the Euler equations is not only an interesting curiosity; it also suggests that well established results in fluid dynamics can be advantageously translated into the beam framework and help us understand and predict beam dynamics in cyclotrons under the effect of space charges.

Scaling with Beam Density

The first thing to observe is that δ^2 only appears in Eqs. 21 and 22 hidden in the time parameter t_2. That means that for a given magnetic field, beams with different densities (or equivalently different currents) will evolve exactly in the same way, only at a different rate. In other words, the phenomena will be identical, but the growth rates grow in time like δ^2, i.e. they scale linearly with the beam density (i.e. the beam current). This is precisely what is observed during beam breakup in the Small Isochronous Ring (SIR) at Michigan State University [10]: the number of clusters in which the beam breaks is independent of the beam current, but the growth rate scales linearly with the beam current. We will shortly return to beam breakup within the framework of our model. First, we discuss the stability of round beams.

Round Beams

By choosing polar coordinates (r, θ) centered at the beam centroid, one can see that an initially round density distribution, i.e. satisfying $\bar{n}_0(\mathbf{r}, 0) = \bar{n}_0(r)$, trivially satisfies Eqs. 21 and 22 and thus remains round for all time.

A more interesting question to ask is what happens to round density distributions to which one adds small amplitude perturbations on the surface. From fluid dynamics theory, one knows that if the beam density profile is monotonically decreasing, the surface modes are stable [14] (and references therein). This explains why round beams with such density profiles are invariably found to be long lived in experiments and simulations [1, 12]. The situation is different when the density profile is hollow: the surface perturbations are then Kelvin-Helmholtz unstable [14].

Elliptic Beams

The simplest case to consider for elliptic beams is that of a uniform density distribution with a sharp edge. This case was studied extensively in the context of fluid dynamics [15] (and references therein), in which it is known as a "Kirchhoff vortex". One can show that for any ratio a/b of the semi-major axis a and semi-minor axis b such a vortex rotates with the uniform rotation frequency $\omega = ab/(a+b)^2$. Furthermore, if $a/b < 3$ the ellipse is stable to edge density perturbations, while for $a/b > 3$ the ellipse is unstable to edge density perturbations. As the ratio a/b increases, instability thresholds for perturbations with shorter and shorter wavelengths are reached [15]. Such instabilities, sometimes called unstable diocotron modes [14], can lead to beam breakup [15]. This is what we show in Figure 1, in which we used the numerical method suggested in [15] to study the evolution of an elliptic beam with aspect ratio $a/b = 6.5$ and uniform density, to which one added small surface perturbations. The perturbation corresponds to an $m = 4$ mode, i.e. it has 4 full periods along the circumference of the ellipse, and we chose $\delta^2 = 0.2$ for the simulation.

Note that strictly speaking, it can be shown that Eq. 21 does precisely *not* allow beam breakup, however thin filaments may be. However, under the effect of the beam instability, strong filamentation occurs and the filaments are so thin that the assumptions used to derive Eq. 21 are not justified anymore in these regions. Specifically, viscosity has to be included at these small scales, and one expects viscosity to allow the beam to break up in two pieces that each have a stable aspect ratio $a/b < 3$.

The situation is more complicated for more realistic elliptic beams with smoother density profiles [16] (and references therein). Unlike the Kirchhoff case the rotation is not uniform, and under the effect of differential rotation

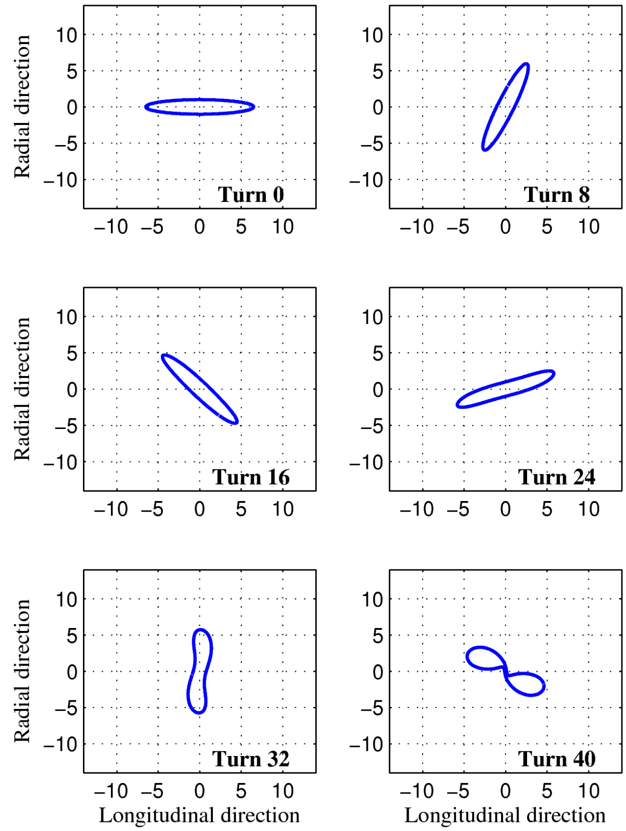

Figure 1: Evolution of a coasting elliptic beam with uniform density as determined by Eqs. 21 and 22. The beam's transport direction is in the negative x direction, the aspect ratio of the ellipse is $a/b = 6.5$, and $\delta^2 = 0.2$. The numerical method used for this simulation is described in [15]. We see that the small $m = 4$ perturbation grows and leads to beam breakup at turn 40.

such beams develop low density filaments on the exterior of the beam, even when $a/b < 3$. These filaments gradually wrap around the core of the beam, and interactions between the filaments and the beam core lead to axisymmetrization of the entire beam. This phenomenon is well understood in the context of fluid dynamics, and has been shown to be robust for reasonably smooth profiles [17]. Unsurprisingly, this is what we obtain in numerical simulations, as we show in Figure 2, which we obtained for $\delta^2 = 0.2$ and a Gaussian density distribution of the form $n(x, y) = \exp(-x^2/2\sigma_x^2 - y^2/2\sigma_y^2)$ with $2\sigma_x = 9$ and $2\sigma_y = 3.5$. The last plot in Figure 2 shows the formation of a stable round beam core and a low density halo coming from the filaments that have separated from the core because of numerical viscosity. This is in agreement with previous PIC simulations of analogous situations [1, 4]. We showed elsewhere [12] that for the same beam parameters as in [1] and [4], we obtain very good agreement with these simulations. This is the sign that the details of the magnetic geometry and three-dimensional effects do not play a significant role, and the essential elements of the dynamics are contained in Eqs. 21 and 22.

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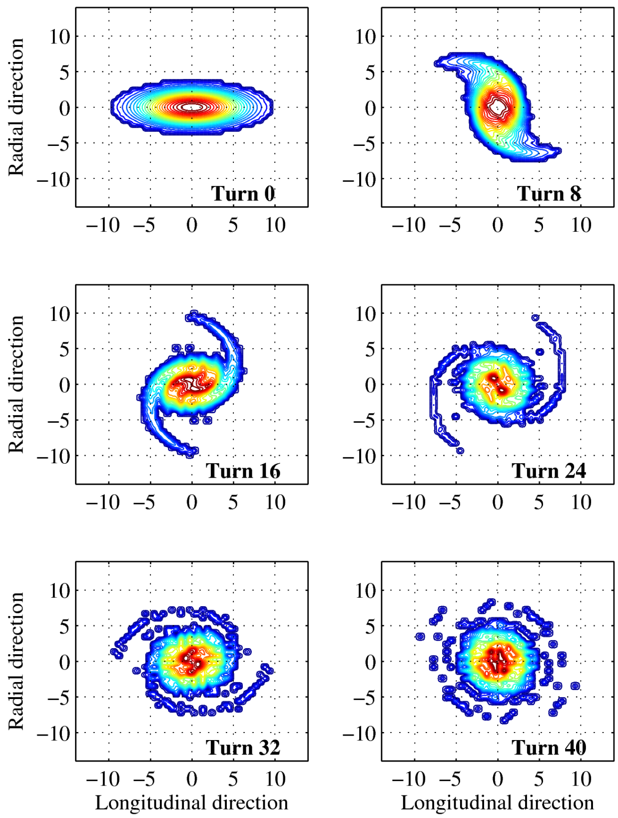

Figure 2: Evolution of a coasting elliptic beam with the Gaussian density profile given in the main text, as determined by Eqs. 21 and 22. The beam's transport direction is in the negative x direction, $\delta^2 = 0.2$, and the dark blue contour line corresponds to 10% of the maximum density. The numerical method used for this simulation is described in [12]. We see the formation of a round beam core and a low density halo after about 40 turns.

CONCLUSION – FUTURE WORK

In many cyclotrons, the time scale associated with betatron oscillations is much shorter than the time scale associated with space charge effects. We have used an averaging procedure based on this fact to derive two simple fluid-like equations for the radial-longitudinal dynamics of a coasting quasi-laminar beam under the effect of space charges in a uniform magnetic field. These equations describe the advection of the beam in the $\mathbf{E} \times \mathbf{B}$ velocity field, where $\mathbf{E}$ is the self electric field. Even though the bunch has a finite temperature, pressure terms do not appear in the equations.

The equations describing the vortex motion are isomorphic to the two-dimensional Euler equations. This means that known results in fluid dynamics on the behavior of isolated vortices can be directly interpreted in the language of beam dynamics in cyclotrons. In particular, round beams with monotonically decreasing density profiles are stable to finite perturbations. Elliptic beams with smooth, monotonically decreasing density profiles are subject to spiraling and "axisymmetrization" [17]. Elliptic bunches with too high aspect ratios break into smaller bunches due to Rayleigh's inviscid shear instability. All these known results are confirmed by our numerical simulations, which are in good agreement with PIC simulations.

In future work, we will consider the role of accelerating gaps in this vortex motion. It might indeed be possible, when desirable, to tailor the accelerating electric field in such a way that it counters the natural spiraling of elongated beams. We will also allow large departures from the laminar regime. The amplitude of the betatron oscillations then determines the size of the bunch, and the problem has to be treated with kinetic equations in phase space instead of the fluid-like equations derived in this work. Solutions will have to be obtained from numerical simulations, but we still expect the averaging procedure to lead to substantial computational savings. Eventually we plan on including three-dimensional effects and exact magnetic field geometries.

REFERENCES

[1] J.J. Yang, A. Adelmann, M. Humbel, M. Seidel, and T.J. Zhang, Phys. Rev. ST Accel. Beams 13 (2010) 062401

[2] A. Adelmann, C. Kraus, Y. Ineichen, S. Russel, and J. Yang, Paul Scherrer Institut Technical Report No. PSI-PR-08-02, 2008

[3] A. Adelmann, P. Arbenz, and Y. Ineichen, J. Comp. Phys 229 (2010) 4554-4566

[4] S. Adam, in Proceedings of the 14th International Conference on Cyclotrons and their Applications, edited by J.C. Cornell (World Scientific, Singapore, 1995), p.446

[5] F.I. Parra and P.J. Catto, Plasma Phys. Control. Fusion 50 (2008) 065014

[6] M. Kotschenreuther, G.W. Rewoldt, and W.M. Tang, Comput. Phys. Commun. 88 (1995) 128

[7] F. Jenko, W. Dorland, M. Kotschenreuther, and B.N. Rogers, Phys. Plasmas 7 (2000) 1904

[8] W. Dorland *et al.*, URL Gyrokinetic Simulations Project, `http://gyrokinetics.sourceforge.net/`, (2009)

[9] J. Candy and R. Waltz, J. Comput. Phys. 186 (2003) 545

[10] E. Pozdeyev, J.A. Rodriguez, F. Marti, and R.C. York, Phys. Rev. ST Accel. Beams 12 (2009) 054202

[11] R.C. Davidson, *Physics of Nonneutral Plasmas* (Addison-Wesley, Reading, MA, 1990)

[12] A.J. Cerfon, J.P. Freidberg, F.I. Parra and T.A. Antaya, Phys. Rev. ST Accel. Beams 16 (2013) 024202

[13] M.M. Gordon, in Proceedings of the 5th International Cyclotron Conference (Oxford, United Kingdom, 1969), p.305

[14] C.F. Driscoll and K.S. Fine, Phys. Fluids B 2 (1990) 1359

[15] T.B. Mitchell and L.F. Rossi, Phys. Fluids 20 (2008) 054103

[16] D.G. Dritschel, J. Fluid Mech. 371 (1998) 141

[17] M.V. Melander, J.C. McWilliams and N.J. Zabusky, J. Fluid Mech. 178 (1987) 137