SPACE CHARGE LIMIT IN SEPARATED TURN CYCLOTRONS*

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Abstract

From both simulations and measurements, it is known that at sufficiently high charge per bunch, the bunches in an isochronous cyclotron undergo a vortex effect whose ultimate result is that the bunches reshape into circularly-symmetric distributions in the radial-longitudinal plane. This state cannot exist for arbitrarily high charge since at some point the space charge force will overwhelm the cyclotron's radial magnetic focusing. We apply envelope equation (or "second moment") formalism to determine (a) the particle motion frequencies (b) the self-consistent size, or turn width, and (c) the upper limit for the bunch charge for a given size of the bunch. This work is partly a review of work by Sacherer, Kleeven, and Bertrand-Ricaud, and partly a synthesis of those works. Some comparisons are made to published data for the PSI high intensity cyclotrons and new data from the TRIUMF cyclotron.

VORTEX EFFECT, QUALITATIVELY

The basic physics of the "vortex" effect is that leading particles are "pushed" by space charge, but cannot advance because of isochronism and instead gain energy and so go sideways to higher radius. Trailing particles do the reverse. Particles at the outside move back and those at the inside move forward. Mort Gordon [1] made the perceptive connection to Coriolis force in rotating frames. He made the following points in regards to space charge forces in cyclotrons. Referenced to the rotating frame in which the bunch is stationary, the motion of the particles due to space charge is a *steady-state velocity ... directed along the equipotential curves associated with F [the electric field due to space charge]*. Gordon realized that this motion applies locally to individual turns, but then since he had dealt only with cyclotrons with little or no turn separation, stated: *Since the length $R\Delta\theta$ of the turn is generally much greater than the radius gain per turn, the local vortices are so small and feeble that their presence can be neglected entirely*. But in high intensity machines, turn separation is required all the way out to extraction, so Gordon's case does not apply and contrarily, the local vortices dominate. Moreover, Gordon probably did not realize that the local vortex effect can impede the usual bunch stretching where R increases and $\Delta\theta$ remains constant: above a threshold amount of bunch charge, $\Delta\theta$ will decrease rather than remain constant, in order to maintain the circular stationary state.

A crucial ingredient in the physics of the evolution of the vortex effect is that the equipotentials of space charge are different than the distribution density contours. Specifically, for $a,\ b$ as semi-axes, the form of the equipotentials is

$$\text{constant} = \frac{x^2}{a(a+b)} + \frac{y^2}{b(a+b)}, \tag{1}$$

when the distribution has the form

$$\text{constant} = \frac{x^2}{a^2} + \frac{y^2}{b^2}. \tag{2}$$

And in fact the two only agree when the distribution is circular.

In 1981, Werner Joho [2] presented a model that used Gordon's mixed turns with constant $R\Delta\theta$. This results in a distribution of charge that looks like a pie sector, and was therefore called the "sector model". From this, he derived a limit beyond which the additional energy spread due to space charge would cause turns to no longer be sufficiently separated for clean extraction. The result was a current limit proportional to the cube of the energy gain per turn. The (local) vortex effect was subsequently discovered at PSI and although it invalidates the sector model, the limit nevertheless still is proportional to the cube of the energy gain per turn. We shall see why.

SIMPLIFIED MODEL

Bertrand and Ricaud [3] have presented an elegant simplified model, from which they derive the relation between bunch charge and size. Since it is known that the bunches tend to circular, for the case where the vertical size is equal to the horizontal (often a fairly good approximation), the bunches are spheres. Spheres with constant inside density of charge have a very simple form for the electric field:

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^3}\,\vec{r} \equiv k\,\vec{r} \tag{3}$$

Further, we assume a flat magnetic field B. Then the magnetic and electric forces on a particle of charge q, mass m give, in the lab frame:

$$\begin{aligned} m\ddot{x} &= +qB\dot{y} + qk(x-x_0) \\ m\ddot{y} &= -qB\dot{x} + qk(y-y_0) \end{aligned}$$

where $(x_0, y_0) = R(\cos\omega t, \sin\omega t)$ is the equilibrium orbit and $\omega = qB/m$.

Solve using complex $z = x + iy$, let $z = R\exp(i\omega t) + C\exp(pt)$, find

$$p^2 - i\omega p - \frac{qk}{m} = 0 \rightarrow p = \frac{i\omega}{2} \pm \sqrt{-\frac{\omega^2}{4} + \frac{qk}{m}} \tag{4}$$

Divide p by $i\omega$ to get the tunes of the modes:

$$\nu_{r\pm} = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{Q}{Q_{\max}}}\right) \tag{5}$$

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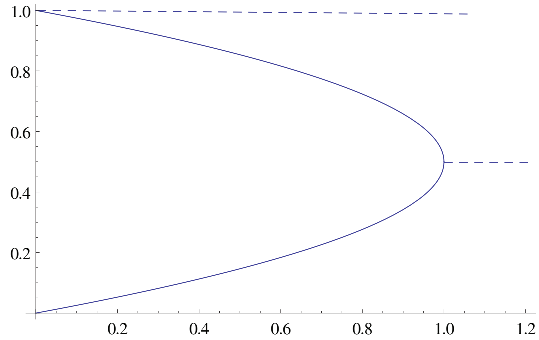

Figure 1: Mode tunes $\nu_{r\pm}$ vs. $Q/Q_{\max}$. In the lab frame, the low mode is the betatron motion and the high mode is the dispersion motion. In the bunch's rotating frame, the modes are the same but roles are reversed. For $Q/Q_{\max} > 1$, there are no stable modes.

where

$$Q_{\max} = \pi\epsilon_0 \left(\frac{m}{q}\right) \omega^2 r^3 \tag{6}$$

For $Q > Q_{\max}$, p has a real part giving exponentially growing solutions. As we approach this limit from below, the acceptance approaches zero; at the limit the beam must have zero emittance in the horizontal plane.

For $Q \ll Q_{\max}$, we find the tune shift.

$$\Delta\nu_r = -\frac{Q}{4Q_{\max}} = -\frac{NR^2 r_p}{\beta^2 r^3} \tag{7}$$

(The rightmost here is written in the form of the "Laslett tune shift". Note "bunching factor" $B_f \propto r/R$. $\Delta\nu_r$ is actually 1/2 the "Laslett space charge tune shift", as the full shift is shared between radial and longitudinal.)

Here's a simpler formula for maximum charge: Notice $\omega = c/R_\infty$, $mc^2/q \equiv V_m$ (938 MV for protons), $\epsilon_0 = (cZ_0)^{-1}$, where $Z_0 = 377\,\Omega$:

$$Q_{\max} = \pi \left(\frac{V_m}{cZ_0}\right) \frac{r^3}{R_\infty^2} \tag{8}$$

Example: The PSI Injector 2: $V_m = 938\,\text{MV}$, $R_\infty = 9.54\,\text{m}$, $r = 6.5\,\text{mm}$. This yields $Q_{\max} = 78\,\text{pC}$; multiply by rf frequency of 50 MHz, we get $I_{\max} = 4\,\text{mA}$. This must be taken in the context of the approximations made: field is still flat, it is non-relativistic, bunches are spheres.

However, we have established that: $Q_{\max} \propto r^3$. Since the rf voltage needed for clean extraction is $V_{\text{rf}} \propto r$, we have

$$I_{\max} \propto V_{\text{rf}}^3 \propto \text{turns}^{-3}, \tag{9}$$

in agreement with PSI's oft-quoted "scaling law".

In the realistic case when Q exceeds $Q_{\max}$, the beam size r simply increases to raise $Q_{\max}$. Bertrand and Ricaud [3] go on to the self-consistent case of a spherical bunch with fixed emittance rather than fixed size. They derive the following quartic equation.

$$r^4 - C\,r - r_0^4 = 0 \tag{10}$$

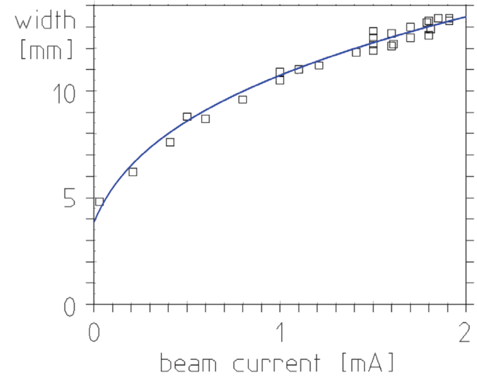

Figure 2: PSI Injector 2 beam size versus current (squares, after Stammbach et al. [4]), and fitted curve from quartic Eq. 10.

where $C = \frac{qQ}{\pi\epsilon_0 m\omega^2}$, $r_0 = \sqrt{2R\epsilon_x}$.

At zero charge, one would expect $r_0 = \sqrt{R\epsilon_x}$ since the Courant-Snyder β-function $\beta_x = R$ for this flat magnet. However, there is a factor of 2 arising from the fact that circular bunches can only be stationary if the longitudinal and radial emittances are equal. Since the beam must be dispersion-matched, each emittance contributes to the beam size.

Leaving r_0 and C as free parameters, we can fit the PSI Injector 2 beam size versus current plot [4] (Fig. 2). The quality of fit suggests the model is valid.

GENERAL CASE

The more general case (relativistic, non-spherical bunches in non-flat magnetic field) can be analyzed using the envelope formalism of Sacherer [5]. This has already been done by Wiel Kleeven [6] in his 1988 thesis work. This work was the first to prove the stationarity of the circular distribution. Here I capitalize on it to derive the more general quartic equation governing the envelope (beam) size.

It was Sacherer's discovery that *the second moments of any particle distribution depend only on the linear part of the force ..., while this linear part of the force in turn depends only on the second moments of the distribution.* These second moments are simply the elements of the $\boldsymbol{\sigma}$-matrix in the TRANSPORT formalism. Their equation of motion is

$$\boldsymbol{\sigma}' = \mathbf{F}\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{F}^T \tag{11}$$

where $\mathbf{F}$ is the infinitesimal transfer matrix, which includes both the focusing and space charge forces. The $\boldsymbol{\sigma}$-matrix element equations can often be combined to find the second order equation governing the evolution of the root mean squared size. For the decoupled 2D unbunched or DC space charge case, this results in the usual Kapchinsky-Vladimirsky equations [5]. In that case, the space charge term in the second derivative of the rms size comes from the simple form of Eq. 1, where, recursively, a and b are the rms sizes.

Space Charge in 3D

In the general 3D bunched beam case, Eq. 11 represents 21 first order equations of the 21 independent second moments. For any given case, it is straightforward to numerically integrate these with, for example, a standard Runge-Kutta code. The linear part of the space charge force is not as simple as in 2D, depending upon elliptic integrals of a particular type called a Carlson symmetric form. The computer code TRANSOPTR [7] uses this technique to calculate 3D envelopes of a bunched beam. We have used it to design the new 300 keV H^- injection line vertical section, where strong bunching results in large 3D space charge, and there is strong coupling among all 3 planes [8].

Kleeven [6] used the Sacherer technique in two stages. First he derived the 21 second-moment equations in normalized coordinates and demonstrated that the stationary distribution is a round beam. Specifically, he derives the following conditions for stationarity [6, Eq. 4.84]:

$$\langle x^2\rangle = \langle s^2\rangle,\ \langle P_x^2\rangle = \langle P_s^2\rangle, \tag{12}$$

$$\langle xP_x\rangle = \langle sP_s\rangle,\ \langle xs\rangle = \langle P_xP_s\rangle = 0, \tag{13}$$

$$\langle sP_x\rangle = -\langle xP_s\rangle = L/2 \tag{14}$$

(here, x, P_x are the radial phase space coordinates and s, P_s the azimuthal or longitudinal; z is vertical.) L is an invariant; it is the angular momentum of the beam about the vertical axis, and we set it to zero. Under these conditions, the radial and azimuthal emittances are necessarily equal.

Envelope Equations

Secondly, as under these conditions the beam evolution again reduces to one with two uncoupled 2D subspaces as in the KV case, Kleeven simplified the 21 equations into two second order envelope equations whose coupling is only through the dependence of the space charge force on the dimensions of the bunch. I rewrite his Eqs. 4.85 in the following form:

$$r'' + \frac{\nu_x^2}{4R^2}\,r - \frac{\epsilon_x^2}{r^3} - \left(\frac{QcZ_0}{4\pi V_m\beta^2\gamma^2}\right)\frac{g_r}{r^2} = 0 \tag{15}$$

$$\zeta'' + \frac{\nu_z^2}{R^2}\,\zeta - \frac{\epsilon_z^2}{\zeta^3} - \left(\frac{QcZ_0}{4\pi V_m\beta^2\gamma^2}\right)\frac{g_z}{\zeta^2} = 0 \tag{16}$$

$r = \sqrt{5\langle x^2\rangle} = \sqrt{5\langle s^2\rangle}$, $\zeta = \sqrt{5\langle z^2\rangle}$, and emittances ($\epsilon_x$, ϵ_z) are 5 times the corresponding rms emittances. The form factors (g_r, g_z) are the Carlson elliptic integrals R_D:

$$g_r = R_D(1, \frac{\zeta^2}{r^2}, 1) = \frac{3}{2}\int_0^\infty \frac{ds}{(1+s)^2(\frac{\zeta^2}{r^2}+s)^{1/2}} \tag{17}$$

$$g_z = R_D(\frac{r^2}{\zeta^2}, \frac{r^2}{\zeta^2}, 1) = \frac{3}{2}\int_0^\infty \frac{ds}{(1+s)^{3/2}(\frac{r^2}{\zeta^2}+s)} \tag{18}$$

Interestingly, these envelope equations are identical to Sacherer's envelope equation for bunched beam transport [5], but the equation for r has a focusing term 4 times smaller than would obtain for x were it not coupled to longitudinal.

Quartic Equation

To find the matched case, we set r to a constant:

$$r^4 - C_r\,r - r_0^4 = 0 \tag{19}$$

where

$$C_r = \frac{qQg_r}{\pi\epsilon_0 m\omega^2\nu_x^2\gamma^2} = \frac{QcZ_0R_\infty^2 g_r}{\pi V_m\nu_x^2\gamma^2}, \tag{20}$$

$$r_0 = \sqrt{2\,\frac{R}{\nu_x}\,\epsilon_x}. \tag{21}$$

This generalizes the Bertrand-Ricaud equation [3]: we see that the two agree when $\nu_x = \gamma = g_r = 1$.

For the vertical case, the quartic equation has the same form ($g_z,\ \nu_z$ replace $g_r,\ \nu_x$), but the coefficients are 4 times smaller:

$$\zeta^4 - C_z\,\zeta - \zeta_0^4 = 0 \tag{22}$$

where

$$C_z = \frac{qQg_z}{4\pi\epsilon_0 m\omega^2\nu_z^2\gamma^2} = \frac{QcZ_0R_\infty^2 g_z}{4\pi V_m\nu_z^2\gamma^2}, \tag{23}$$

$$\zeta_0^2 = \frac{R}{\nu_z}\,\epsilon_z. \tag{24}$$

Because of this factor of 4, the space charge limit for horizontal motion is usually lower than for vertical.

Other Distributions

It is worth emphasizing that the quartic equation is still correct for non-hard-edge bunches and even for non-ellipsoids. This was proved by Sacherer [5]. In those cases, r is no longer the radius of the hard-edged sphere, but is $\sqrt{5}$ times the rms size, and the emittance is 5 times the rms emittance. If the bunch shape is far from stationary, then it will change with time and so the rms emittance also will change with time. In that case, the envelope equations, while still correct, are not as useful.

INTENSITY LIMIT

If we know the maximum allowed rms width of a turn, we can find the maximum charge per bunch from ($r_0 = 0$)

$$C_{r\max} = r_{\max}^3. \tag{25}$$

The formula is thus

$$Q_{\max} = \frac{\pi}{g_r}\left(\frac{V_m}{cZ_0}\right)\frac{r^3}{R_\infty^2}\nu_x^2\gamma^2 \tag{26}$$

The maximum allowed size to cleanly extract ($r_{\rm f}$) is some factor, say ξ smaller than the radius gain per turn at extraction. Using the well-known formula for turn separation,

$$\xi r_{\rm f} = \frac{R_\infty}{\beta_{\rm f}\gamma_{\rm f}\nu_{x,{\rm f}}^2}\frac{V_{\rm rf,f}}{V_m} \tag{27}$$

where $V_{\rm rf,f}$ is the rf voltage per turn on the final orbit and $V_m = mc^2/q$.

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Let h be the number of bunches per turn, convert the charge per bunch to current $I = \frac{hQc}{2\pi R_\infty}$, we find a simple expression for maximum current

$$I_{\max} = \frac{h}{2g_r\xi^3\beta^3\gamma\nu_x^4}\frac{V_{\rm rf}^3}{V_m^2 Z_0} \qquad (28)$$

where g_r, β, γ, ν_x, and $V_{\rm rf}$ have of course their extraction values. Note: for aspect ratios in the range $1/2 \le \zeta/r \le 2$, the approximation $g_r \approx 1 - \frac{3}{5}\log(\zeta/r)$ works well.

Let us assume spherical bunch approximation and take $\xi = 2.7$; this means the allowed turn width is $2.7\sqrt{5} = 6$ times the rms size. Then for the PSI Ring (590 MeV, $h = 6$, $V_{\rm rf} = 3$ MV) and the PSI Injector 2 (72 MeV, $h = 10$, $V_{\rm rf} = 0.75$ MV) we find respectively 2.2 mA and 2.1 mA. These can be seen to be quite good, recalling that the sector model fails by about an order of magnitude for PSI Injector 2 [9]. Since the formula 28 follows from the extreme case of zero emittance (causing it to overestimate) and, on the other hand, it does not account for other possible tricks for increasing turn separation at extraction such as coherent oscillations (causing it to underestimate), it can be expected to be within a factor of only ~ 2 of the real limit.

However, the formula is useful for scaling. We have the "cubic scaling law" with rf voltage, but further:

- For given energy per nucleon, heavier particles hinder rather than help. The limiting rate of nucleon delivery is $I_{\max}m/q \propto q/m$.
- Large radial tune at extraction hinders rather than helps: it increases the space charge limit for a given beam size, but it reduces radius gain per turn and the latter effect dominates.
- Higher magnetic field scales down the machine size. In principle, this neither helps nor hinders, however if electric fields are limiting, a smaller machine cannot maintain the same rf voltage so $I_{\max}$ would be lower.
- More bunches per turn always helps. But higher harmonic number may cause difficulty at injection.

ACCELERATION EFFECTS

A surprising consequence of the vortex effect is that the bunches remain the same length, or even decrease in length (for relativistic energies) as they accelerate, thus decreasing in phase length. For circular bunches, the energy spread $\Delta E \sim \beta\gamma^3\Delta\beta = \beta\gamma^3\Delta R/R_\infty$ is tied to the betatron width through $\Delta R = \sqrt{\epsilon_n R_\infty}/\gamma \propto \gamma^{-1}$. Thus $\Delta E \propto \beta\gamma^2$ and since $\Delta E\Delta t$ phase space area is invariant, $\Delta t \propto (\beta\gamma^2)^{-1}$, or bunch "length"$\propto \gamma^{-2}$. Note that this means that at relativistic energies, the bunches are not circular but have an aspect ratio $\Delta R/\Delta s = \gamma$. This can be considered as due to relativistic "length contraction". (Nevertheless, the bunches are circular in their own rest frame, so the Carlson integral arguments are not modified by this aspect ratio [10].)

Another remarkable feature of the Kleeven distribution is that it is independent of intensity; the stationary state is a circular bunch at any finite bunch charge. This seems a paradox, since we know that at low intensity, particles have invariant phase, thus maintaining bunch time spread Δt: physically, acceleration stretches the bunches such that the set of neighbouring turns appears as a pie sector. (Hence, the Joho "sector model" [2].) There is an fact a competition between two effects: if the rate of movement within a bunch due to space charge is large compared with the stretching rate, then the bunches will remain circular. This circulating rate is simply the space charge tune shift (7). Staying with non-relativistic (where the stretching effect is fastest), we compare: azimuthal change per turn from acceleration $\delta R\Delta\theta$ with radial change per turn from space charge $2\pi\Delta\nu_r R\Delta\theta$. ($\delta R$ is radius gain per turn, $R = \beta R_\infty$, $\Delta\theta$ is azimuthal extent). Thus if

$$2\pi\Delta\nu_r \gg \frac{\delta\beta}{\beta}, \qquad (29)$$

then a launched circular dispersion-matched bunch will remain circular. A non-matched non-circular bunch will match itself after a number of turns $\gg 1/\Delta\nu_r$, and the generated halo will depend upon the initial mismatch.

Importantly, remember that the cyclotron space charge tune shift at non-relativistic energy is independent of energy: $\Delta\nu_r = -\frac{qQ}{4\pi\epsilon_0 r^3 m\omega^2}$, r and ω constant, and that the space charge limit is where the tune is depressed by half. Thus an injector cyclotron operating near the space charge limit will have $\Delta\nu_r \sim \nu_x/2 \sim 1/2$. Since usually in such a machine $\delta\beta \sim \beta$, it will start and remain in a vortex state. ($\delta\beta$ rapidly decreases.) This is the case for PSI Injector 2, already at turn 1.

For TRIUMF, just after the injection gap, energy is 390 keV, and after one turn it is 750 keV, so $\frac{\delta\beta}{\beta} = 0.6$. The tune shift at 290 μA is $\sim 1/40$, so $2\pi\Delta\nu_r \sim 0.16$. So it is clearly in an in-between state: bunches stretch and also have some vortex character.

This is verified when we look at the turn structure of the first ~ 30 turns (see Fig. 3). At low bunch charge, the turn structure is persistent, but at higher charge, the turn separation disappears after 10 turns and reappears at 20 where the core of the bunch has executed a half turn. For more details, see ref. [11]

Perhaps surprisingly, however, this mixed-up state has little consequence for TRIUMF because extraction is by stripping and separated turns are not needed. Irregular turns do not contribute to extracted emittance or energy spread because the distortions happen in such a way as to maintain the close correlation between energy and radius. This allows compact H^- cyclotrons in general to reach currents in the mA range with good beam quality in spite of a messy and chaotic injection process. In separated turn extraction machines, all particles must execute the same number of turns and this puts a very tight constraint on the injection matching, especially the initial phase spread: the injected

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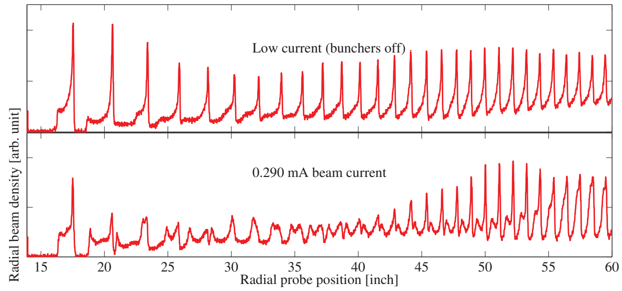

Figure 3: TRIUMF turn structure appears even though the bunches are typically 10 times longer than their width. The sharp peaks occur at the waveform crest; they are the projection of an ArcCos. (Lower) Space charge causes this central core to rotate, thus losing the turn structure at 1/4 of $1/\Delta\nu_r$, and reappear at 1/2. See ref. [11] for further details.

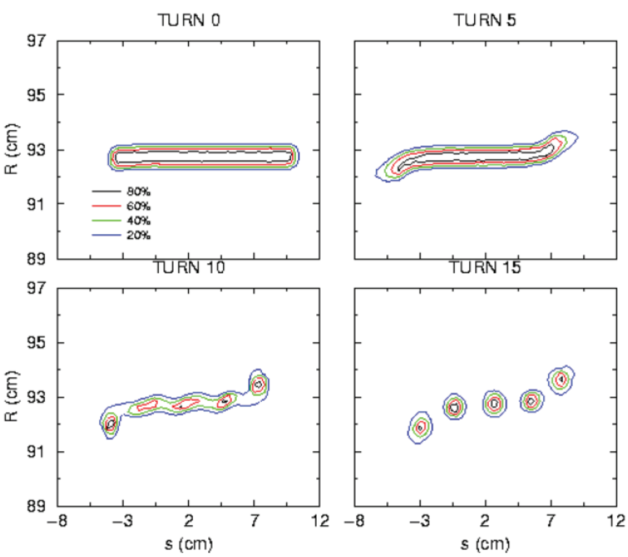

Figure 4: Pozdeyev [12] simulation for SIR. Long bunch splits into small circular droplets; number of droplets $\sim$ Length$/(2 \times$ width$)$.

bunch must have a length not much longer than the turn width.

In masterful work for his Ph.D. thesis, Eduard Pozdeyev [12] has shown, both in simulation and in experiments in MSU's Small Isochronous Ring (SIR) how a bunch evolves if its length is large compared with its width. The bunch splits into droplets, each of which has the expected circular distribution, and the number of droplets is given by the aspect ratio.

For the case of injecting into a cyclotron and subsequent acceleration, the droplets resulting from a long bunch would be arranged as beads on a parabolic necklace in R vs. θ. This would prohibit separated turn extraction. Thus, the remaining path to higher intensity is higher harmonic rather than longer bunches. In a way similar to the competition between stretching and turning, space charge, if sufficiently strong can also mitigate the effect of nonlinearity of the rf waveform. The condition is:

$$2\pi\Delta\nu_r \gg \frac{\text{turn separation}}{\text{bunch length}}\frac{(\Delta\phi)^2}{2} = \frac{\delta\beta}{\beta}\frac{h\Delta\phi}{2} \qquad (30)$$

($\Delta\phi = h\Delta\theta$ is rf phase length.) This condition is in fact met in the PSI Injector 2, and explains why they do not need third harmonic flat-topping at their highest intensity.

CONCLUSIONS

A new formula for maximum intensity from separated turn cyclotrons has been derived from envelope theory, and its scaling characteristics explored. The formula applies to cases where the injected bunch is sufficiently short that the vortex effect curls it up into a single droplet. A qualitative intensity threshold has been derived for the vortex effect to take place; below the threshold, bunches expand to maintain their phase length as β increases, but above it, the bunch maintains length and consequently decreases in phase length. Also, at sufficiently high current, the vortex effect dominates over rf waveform nonlinearity, obviating need for flat-topping.

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