OPTIMUM SERPENTINE ACCELERATION IN SCALING FFAG

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Abstract

Serpentine acceleration is typified by fixed radio frequency, fixed magnetic field and a near (but not) isochronous lattice, radial motion of the orbit, and two or more reversals of the motion in RF phase. This was discovered[1] in 2003 for linear non-scaling FFAGs in the relativistic regime. In 2013, Kyoto University School of Engineering[2] pointed out that serpentine acceleration is possible also in scaling FFAGs and may span the non-relativistic to relativistic regime. As a function of two key parameters, field index and synchronous energy, this paper shows how to optimize the extraction energy and the voltage per turn for the scaling case. Optimization is difficult, and typically leads to poor performance: either extreme voltage or small acceleration range. Nevertheless, designs with credible acceleration parameters can be obtained; and indicative examples are presented herein.

THEORY

Let us contrast the FFAG against the synchrotron. In the latter, the properties of a general particle w.r.t. synchronous are kept (almost) constant by ramping the magnetic field. The "synchronous energy" is a function of time and there is a single orbit. The motion about this is given in power series expansion in small quantities in the longitudinal coordinates momentum P or total energy E. Contrastingly, in a scaling FFAG, any orbit and energy can be used to define the synchronous reference. Due to the remarkable properties of the magnet lattice, the general particle motion can be written in absolute coordinates. In other words, because the momentum compaction is a global property of the lattice, independent of any selected reference energy, we have no need of power series expansions in small deviations.

In the scaling FFAG, the magnet field has the form:

$$B_z(R, z = 0) = (R/R_0)^k \tag{1}$$

where $k > 0$ is the field index. The general orbit radius is given by $R/R_s = (P/P_s)^\alpha$ where $\alpha = 1/(1 + k) < 1$ is solely a property of the lattice. It follows that revolution period T is given by

$$T/T_s = (E/E_s)(P/P_s)^{(-1+\alpha)} = (\beta_s/\beta)[(\beta\gamma)/(\beta_s\gamma_s)]^\alpha . \tag{2}$$

Here γ is the relativistic kinematic factor, $E = E_0\gamma$ and $E_0 = m_0c^2$ is the rest mass energy. We define $T \equiv T(\gamma)$, $T_s \equiv T(\gamma_s)$ and $T_t \equiv T(\gamma_t)$ where $E_s = E_0\gamma_s$ is a synchronous energy and $E_t = E_0\gamma_t$ is the transition energy. One may eliminate $\beta = v/c$ in favour of γ. As the basis for estimations, useful approximations (in the limit $\gamma \gg \gamma_s$ and $\alpha \ll 1$) are $T/T_s \approx \beta_s[\gamma/(\beta_s\gamma_s)]^\alpha$; we set equal T/T_s to unity and solve for $\gamma \approx \gamma_s/\beta_s^k$.

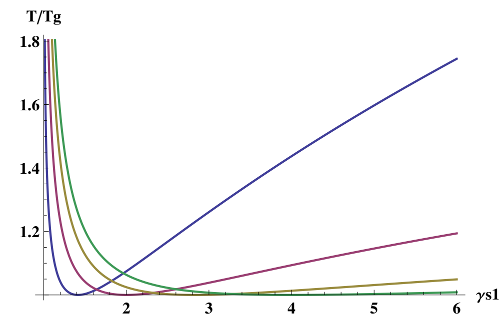

Figure 1: Revolution period versus energy (γ) for $\alpha = 1/2$ (blue), 1/4 (red), 1/8 (yellow), 1/16 (green).

Figure 1 shows T/T_g curves as a function of γ_s for a variety of α. The curves are "U" or "V"-shaped. $\gamma(T)$ is a double valued function: to each value of T belongs two values of γ. Each curve have a minimum which defines the transition energy. Solving $\partial(T/T_s)/\partial\gamma = 0$, one finds $\gamma_t = 1/\sqrt{\alpha}$. All normalized curves $T/T_g = (T/T_s)/(T_g/T_s)$ for a particular value of α have the identical shape, independent of γ_s.

For energy less than the transition value, the revolution period behaves as if dominated by changing particle speed; and above transition behaves as if dominated by path length. If k is sufficiently large and γ_t sufficiently high, this apparent behaviour persists even in the relativistic regime. The difficulty is of course getting k sufficiently large without compromising the transverse optics. Contrastingly, using moderate values of field index produces a machine which can in principle cover the transition from non-relativistic to relativistic - with constant RF.

For brevity, let $\gamma_{s1} \equiv \gamma_1$ and $\gamma_{s2} \equiv \gamma_2$ be two energies having the same revolution period; there is a continuum of such doublets. We shall adhere to the convention that $\gamma_1 < \gamma_t < \gamma_2$. A certain doublet is chosen to be the synchronous reference when we set the radio frequency (RF) to be co-periodic with the orbit period $T(\gamma_1) = T(\gamma_2)$. Once this is chosen E_1, E_2 become fixed points of the motion. Both values of the synchronous E_s are equally valid! It is a little arbitrary, but we choose to work with the lower E_{s1} because it exists in the narrow range $1 < \gamma_{s1} < \gamma_t$.

The general features of the T/T_g curves in Fig. 1 are a very steep rise as $\gamma \to 1$, and a long slow ramp for $\gamma \gg \gamma_t$. When selecting reference doublets, this has the consequence that as $\gamma_1 \to 1$, so $\gamma_2 \to \infty$. Thus the range of acceleration is unbounded. But this range is illusory, and corresponds to a linac-like regime with prodigious voltage requirement.

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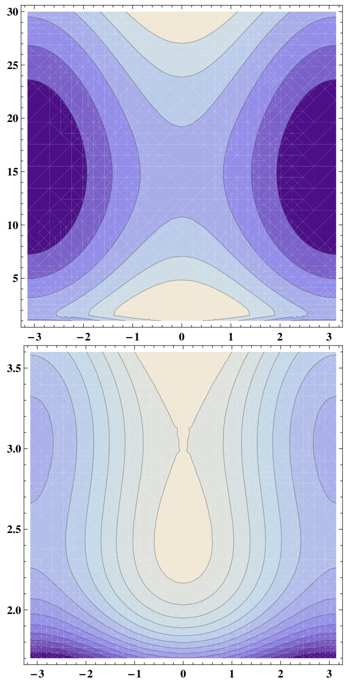

Figure 2: Energy (γ) versus RF phase (ϕ); case 1 (upper) and case 2 (lower)

Hamiltonian

$$H(E,P,\phi) \equiv -Eh + h(PP_s)\frac{(P/P_s)^\alpha}{E_s(1+\alpha)} + \frac{eV\cos\phi}{2\pi}\,. \qquad (3)$$

Because of the FFAG scaling property, the Hamiltonian is invariant whether we use E_1 or E_2 for the synchronous energy. We set h times their common revolution frequency equal to the radio frequency, where h is the harmonic number. These two energies are either side of transition; so, during acceleration, the direction of phase slip for the entire beam reverses twice.

Our task would appear to be to maximize the acceleration range for a given value of the voltage per turn V. Figures 2,5 show examples of phase space contours. Serpentine acceleration in the S-shape channel between two RF buckets offset in energy can be greater than the range (bottom to top) within a single RF bucket. We take the range to be formed of the sum of three phase space arcs: (i) from the injection energy E_i to the first synchronous energy E_1; (ii) a path between E_1 and E_2; and (iii) from the second synchronous energy to the extraction energy E_x.

Minimum Voltage

The condition to connect the two fixed points E_1 and E_2 by a phase space path of zero width is obtained by equating the two Hamiltonians $H(E_1,P_1,\pi) = H(E_2,P_2,0)$ and solving for voltage per turn:

$$\frac{eV_0}{E_0} = \pi h\frac{(\gamma_2-\gamma_1)(\gamma_1\gamma_2\alpha-1)}{\gamma_1\gamma_2(1+\alpha)}\,. \qquad (4)$$

This is a very significant relation. If $\alpha\gamma_1\gamma_2 \gg 1$ this corresponds to acceleration in a linac-like regime (case 1) in which $\Delta\gamma/\gamma_t \gg 1$ and $eV_0/E_0 \to (\gamma_2-\gamma_1)\alpha\pi h/(1+\alpha)$. This is a very few turn acceleration regime, and there is little point employing an FFAG ring unless the particles are very short lived. The required voltage is prodigious: order the rest mass energy per turn; this may be acceptable for leptons but less so for hadrons. Contrastingly, if $\alpha\gamma_1\gamma_2 \to 1$ then $V \to 0$. In principle, this implies $\Delta E/eV \to \infty$; but $\Delta\gamma/\gamma_t \to 0$. This corresponds to acceleration in a ring-like regime (case 3), with tiny voltage and many turns but with a small range. Nevertheless, by fine tuning of parameters, this feature may be exploited to give a limited multi-turn acceleration (cases 2,4,5). $\alpha\gamma_1\gamma_2 = 1$ has the single solution is $\gamma_1\gamma_2 = \gamma_t$. For all other values such that $T(\gamma_1) = T(\gamma_2)$, $\alpha\gamma_1\gamma_2 > 1$ and rises progressively rapidly because γ_2 increases more quickly than γ_1 falls. Clearly, small α is advantageous. The phase acceptance opens from zero to $\cos\phi = [2V_0/V - 1]$.

Acceleration Range

The extraction energy is obtained by equating $H(E_x,P_x,\pi) = H(E_2,P_2,0)$, writing $E_x = E_2 + \delta E_x$, and solving for the increment

$$\delta E_x^2 \approx (2V/\pi h)/[+1/E_2 - E_2/P_2^2(1-\alpha)]$$

The injection energy is obtained by equating $H(E_i,P_i,0) = H(E_1,P_1,\pi)$, writing $E_i = E_1 - \delta E_i$, and solving for the increment

$$\delta E_i^2 \approx (2V/\pi h)/[-1/E_1 + E_1/P_1^2(1-\alpha)]$$

Typically $\delta E_i \ll \delta E_x$: $\delta E_i \sim P_1 c\sqrt{2}$ and $\delta E_x \sim E_2\sqrt{2}$. The energy range of the machine is

$$\Delta E = (E_x - E_i) \approx (E_2 - E_1) + \delta E_i + \delta E_x \sim 2E_2$$

which is expressible solely in terms of E_1, E_2. But E_2 is expressible in terms of E_1: $E_2(E_1)$ is the solution of $T(E_1) = T(E_2)$. Hence there is an expression for the energy range in terms of E_1, V, α.

Figure 3 shows the normalized acceleration range $\Delta E/E_t \equiv \Delta\gamma/\gamma_t$ (red), the voltage eV_0/E_0 (yellow), and $\Delta E/eV_0$ (blue) which is roughly the number of turns, as a function of γ_{s1} for a particular α (case 4). As $\gamma_s \to 1$ the range becomes unbounded; and as $\gamma_s \to \gamma_t$ the range shrinks to zero. The foregoing remarks have prepared us for this. Contrary to expectations, $\Delta E/eV$ is not a suitable figure of merit upon which to base optimization. While $\Delta E/eV$ rises, the acceleration range falls dramatically; the voltage per turn falls even more precipitously. These behaviours are common to all values of α.

OPTIMIZATION

We know that $\gamma_{s1} \to 1$ (large range, large voltage, few turns) and $\gamma_{s1} \to \gamma_t$ (small range, tiny voltage, many turns) are both poor choices for the synchronous energy. But one may speculate that useful working points exit between these extremes. Our approach is to take combinations $[\gamma_1, \gamma_2]$ which satisfy $T(\gamma_1) = T(\gamma_2)$ exactly, and

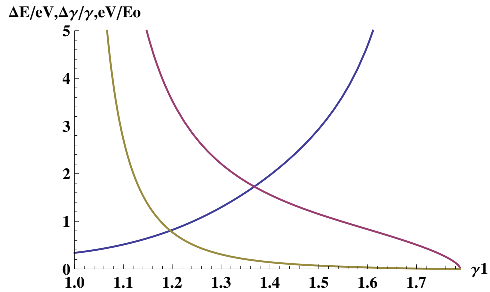

Figure 3: Range per volt $(\Delta E/eV_0)/10$ (blue), total range $\Delta E/E_t$ (red), and minimum voltage eV_0/E_0 (yellow).

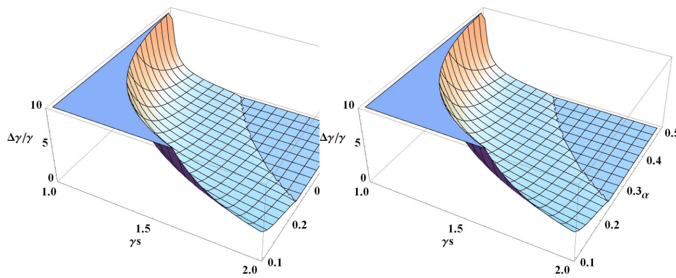

Figure 4: Normalized range (left) and required voltage (right) as function of α, γ_{s1}. Range of $\alpha = [0.1, 0.5]$.

roughly satisfy $\gamma_1\gamma_2 \approx \gamma_t^2$. The optimization amounts to scanning α, γ_{s1}.

Figure 4 shows that maximizing the energy range and minimizing the voltage are contradictory efforts. Thus one must choose, for given index α, either the range and accept the voltage, or place a limit on voltage per turn and accept the energy range. Alternatively, for given range and voltage values one may search for the (α, γ_s) combination that leads to the largest value of α (i.e. smallest value of k) and hence the easiest-to-realize magnetic lattice.

Figure 4 exemplifies the challenge. Let ρ, ν be target values. Optimization corresponds to finding the intersection of the two surfaces: $(\Delta\gamma/\gamma_t)/\rho \geq 1$ and $(eV_0/E_0)/\nu \leq 1$ in the (α, γ_{s1}) plane, subject to greatest α.

Examples

We present five examples, each with different design objectives (ρ, ν). The first case is linac-like, with large range and voltage. The third case is ring-like, with small voltage and many turns. The fifth case is a toy accelerator that spans the Newtonian to relativistic region. The second and fourth cases are intermediate with similar number of turns, but with opposing tendency of α and eV_0/E_0.

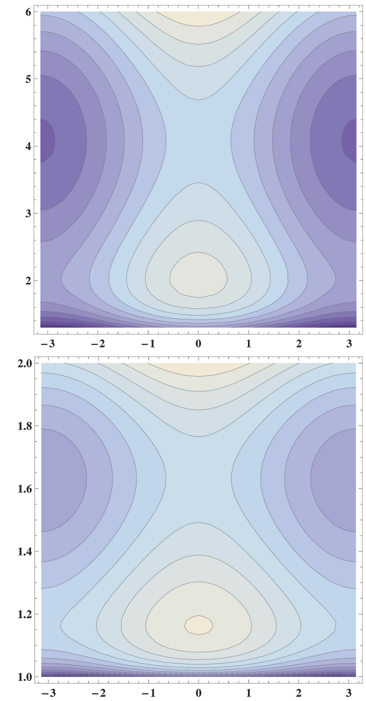

Figure 5: Energy (γ) versus RF phase (ϕ); case 2 (upper) and case 5 (lower).

CONCLUSION

The scaling FFAG proves to be a versatile platform for exploiting serpentine acceleration. However, the performance is generally poor: either the voltages are large and the turns are few, or the voltages and acceleration range are small. In either case, other accelerator types (linac and cyclotron, respectively) would be more effective. Moreover, there is no obvious figure of merit upon which to base optimization. Nevertheless, careful optimization (based on controlling $\alpha\gamma_{s1}\gamma_{s2}$) can produce intermediates cases with credible parameters that have the appeal of acceleration over the Newtonian $(\gamma \approx 1)$ to relativistic regime $(\gamma \gg 1)$. Note, these conclusions do not apply to scaling FFAGs with swept RF; they are a class distinct from the considerations above.

REFERENCES

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[2] E. Yamakawa et al: "Serpentine Acceleration in Zero-Chromatic FFAG Accelerators", Nuclear Instruments and Methods A, Volume 716, 11 July 2013, Pages 4653.

case	α	γ_{s1}	γ_t	$\Delta E/E_t$	$\Delta E/eV_0$	eV_0/E_0	$\Delta\gamma$	γ_{s2}	γ_{inj}
1	0.093567	1.6502	3.2692	10.	16.35	2.0	32.38	14.77	1.0
2	0.134416	2.0249	2.7276	2.0	36.37	0.150	5.462	4.419	1.3061
3	0.137074	2.42703	2.7010	1.0	90.03	0.030	2.699	3.304	1.7683
4	0.312317	1.5461	1.7894	1.0	35.79	0.050	1.7885	2.2788	1.2079
5	0.563233	1.16276	1.3325	0.750	24.98	0.040	1.000	1.64715	1.0269