THE SPIN MOTION CALCULATION USING LIE METHOD IN COLLIDER NONLINEAR MAGNETIC FIELD

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Abstract

Lie operator method of solving the spin motion equation in collider nonlinear fields is used. The matrix presentation of spin Lie transformation for particle passing through collider elements is obtained. The formulas for combined several spin turn transformations are calculated in vector, matrix and operator forms for zero, first and second powers component of dynamical variable vector. The expressions for frequency precession vector components in zero, first and second powers on orbit motion and first powers on spin motion are obtained. The computer codes algorithms for nonlinear spin motion calculation are discussed.

Solution of spin motion

As is known [1], the classical equation of spin motion in the collider is:

$$d\mathbf{S}/ds = [\mathbf{WS}], \tag{1}$$

where $\mathbf{S}$ is a spin vector, s is an azimuth and the precession frequency vector $\mathbf{W}$ is defined by BMT's equation [2]. The equation (1) is written in the frame $(\mathbf{e}_x, \mathbf{e}_z, \mathbf{T})$, fixed relative to the collider.

In the approach, which is based on using the technique of Lie operators, the vectors $\mathbf{S}$ and $\mathbf{W}$ in the equation (1) are considered as operators. Then for a particle with the orbital Hamiltonian H_{orb} and spin Hamiltonian $\mathbf{WS}$ one can find the solution of this equation (the semicolons (":") emphasize the operator nature of the expression):

$$\mathbf{S}(s)=\exp(-:\int_0^s ds'(H_{orb}+\mathbf{WS}):)\mathbf{S}(0). \tag{2}$$

Here, as usual, the exponential operator is understood as a series:

$$\exp(-:F:) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}:F:^n.$$

Each term of this series is the differential operator of n-th power, which action on an arbitrary function f is defined with a help of Poisson brackets:

$$:F:f=\{F,f\} = \frac{dF}{dz_i}\cdot\frac{df}{dp_{zi}} - \frac{dF}{dp_{zi}}\cdot\frac{df}{dz_i}.$$

As is known, Poisson brackets are: $\{q_i,q_k\}=\{p_i,p_k\}=0$, $\{q_i,p_k\}=-\{p_i,q_k\}=\delta_{ik}$, where q,p - are a conjugate dynamical variables pairs (x,p_x), (z,p_z), (σ,p_σ), (J,Φ) and δ_{ik} - is the Kroneker symbol. Since the spin motion equation can't be linearized in spin canonical variables action-angle J, Φ, it is useful to introduce the set of noncanonical variables [3]:

$$S_x=\sqrt{S^2-J^2}\cdot\cos\Phi,\quad S_s=\sqrt{S^2-J^2}\cdot\sin\Phi,\quad S_z=J,$$

here $S^2=S_x{}^2+S_z{}^2+S_s{}^2$. For this set one can find: $\{S_i,S_j\}=e_{ijk}S_k$, where e_{ijk} - is the three-dimensional completely antisymmetric tensor. Besides that for any i,j the following expression takes place: $\{Z_i,S_k\}=\{S_i,Z_k\}=0$.

The operator, which is introduced in this manner is referred to as a Lie operator whereas the exponential series - Lie transformation.

For the total Hamiltonian $H_{orb}+\mathbf{WS}$, which does not depend on azimuth explicitly, one can find instead of (2):

$$\mathbf{S}(s) = \exp(-:s(H_{orb}+\mathbf{WS}):)\ \mathbf{S}(0) = M\ \mathbf{S}(0),$$

where M is a total exponential operator. According to the Hamilton equations, this operator satisfies an equation $^{dM}/_{ds}=M:-(H_{orb}+\mathbf{WS}):$. Let us present this operator as a product of three exponential operators [4,5]: $M=M_2M_1M_0$. To this end expand the total Hamiltonian in a sum of homogeneous polynomials in powers of $\mathbf{Z}$:

$$H_{orb}+\mathbf{WS} = H_2+H_3+H_4+\mathbf{W}_0\mathbf{S}+\mathbf{W}_1\mathbf{S}+\mathbf{W}_2\mathbf{S}+\dots,$$

where subscripts show powers of polynomials. It is important to find of M_2, M_1 and M_0, that operators $:H_2:$ and $:\mathbf{W}_0\mathbf{S}:$ do not change the power of dependence on $\mathbf{Z}$ for any operands, but operators $:H_3:$ and $:\mathbf{W}_1\mathbf{S}:$ increase it by one. Similarly the operators $:H_4:$ and $:\mathbf{W}_2\mathbf{S}:$ increase it by two etc.

Using the Lie technique [6,7] of calculations M_2, M_1 and M_0 one can find, that

$$M=M_2M_1M_0 = \exp(-:f_2:)\cdot\exp(-:f_1:)\cdot\exp(-:f_0:) = (\ E - :f_1: - :f_2: + \frac{:f_1:^2}{2}\ )\ M_0, \tag{3}$$

where

$$:f_0: = :h_2+\mathbf{w}^0\mathbf{S}:= s:H_2+\mathbf{W}_0\mathbf{S}:\ ,$$

operator M_0 is

$$M_0=\exp(-:f_0:)=\exp(:\mathbf{w}^0\mathbf{S}:)\exp(:h_2:)=S\ A$$

and

$$:f_1: = :h_3+\mathbf{w}^1\mathbf{S}:=-:(\int_0^s ds' M_0(s')(H_3+\mathbf{W}_1\mathbf{S})\ ):\ ,$$

$$:f_2: = :h_4+\mathbf{w}^2\mathbf{S}:=-:(\int_0^s ds' M_0(s')(H_4+\mathbf{W}_2\mathbf{S})\ ): -$$

$$-\frac{1}{2}:(\int_0^s ds'\int_0^{s'} ds''[M_0(s'')(H_3+\mathbf{W}_1\mathbf{S}), M_0(s')(H_3+\mathbf{W}_1\mathbf{S})]):.$$

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In this formulas E is a unit operator, M_0 is a usual spin and orbital matrixes S and A of linear transformation and $[\,,]$ - the commutator of operators. Functions h_i and $\mathbf{w}^1\mathbf{S}$ are "integrated" on azimuth polinomials with power equals i on components of $\mathbf{Z}$ for orbital hamiltonian H_i and spin hamiltonian $\mathbf{W}_i\mathbf{S}$.

For calculation one need to know the series of orbital Hamiltonian and spin precession frequency $\mathbf{W}$ (sound from the BMT equation [7]), which are presented in supplement.

The operators for different elements may be grouped in one operator [7]. Therefore spin dynamical characteristics may be investigated with using operators of one elements or group of elements (one ore some period of collider in particularly).

Polarization calculation

The degree of the equilibrium polarization is given by the Derbenev-Kondratenko formula [8]:

$$P_{eq}=\frac{8}{5\sqrt{3}}\,\frac{\alpha_-}{\alpha_+},$$

$$\alpha_- = \int_L ds\; e_z\,(\mathbf{n}-Y\frac{d\mathbf{n}}{dY})\;K^3,$$

$$\alpha_+ =\int_L ds\;[1-\frac{2}{9}(\mathbf{n}e_T)^2+\frac{11}{18}(Y\frac{d\mathbf{n}}{dY})^2]\;K^3.$$

Let us write series $\mathbf{n}$ in power on orbital vector $\mathbf{Z}$:

$$\mathbf{n}_i(s)=\mathbf{n}^o{}_i(s)+\mathbf{n}^1{}_{ip}(s)Z_p(s)+\mathbf{n}^1{}_{ipr}(s)Z_p(s)Z_r(s).$$

Using this series one can find the next transformation formulas for any spin vectors:

$$\mathbf{n}(s)=M\;\mathbf{n}(o)\;,$$

where operator M is determined in (3). Let us rewrite the f_1 and f_2 as coefficients of polinomials:

$$f_1=h^1{}_{pqr}Z_pZ_qZ_r+w^1{}_{ip}Z_pS_i\;,$$
$$f_2=h^2{}_{pqrs}Z_pZ_qZ_rZ_s+w^2{}_{ipr}Z_pZ_rS_i\;.$$

For illustration let us find the rezults of action $:f_1:$ on $\mathbf{Z}$ and $\mathbf{S}$. The Poisson brackets are:

$$:h_1:S_i=0\;,$$
$$:h_1:Z_i=h_{pqr}\{Z_pZ_qZ_r,Z_i\}=3h_{pqr}Z_pZ_q\{Z_r,Z_i\}\;,$$
$$:w^1{}_kS_k:Z_i=\{w^1{}_{kp}Z_pS_k,Z_i\}=w^1{}_{kp}S_k\{Z_p,Z_i\},$$
$$:w^1{}_kS_k:S_i=\{w^1{}_{kp}Z_pS_k,S_i\}=w^1{}_{kp}Z_pe_{kij}S_j\;.$$

Let us introduce some useful definitions:

$$e_{ijk}w^1{}_{jp}\equiv U_{ikp},\; e_{ijk}w^2{}_{jpr}\equiv V_{ikpr},$$
$$3h_{pqr}\{Z_r,Z_s\}\equiv B_{spq}.$$

and separate its on power $\mathbf{Z}$:

$$n^o{}_i(s)=S_{ij}n^o{}_j(o)\;,$$
$$n^1{}_{ip}(s)Z_p(s)=\{S_{ij}n^1{}_{jr}(o)A_{rp}+U_{ikp}S_{kn}n^o{}_n(o)\}\cdot Z_p(o),$$
$$n^2{}_{ipr}(s)Z_p(s)Z_r(s)=\{S_{ij}n^2{}_{jqs}(o)A_{qp}A_{sr}+U_{ikp}S_{kl}n^1{}_{lq}(o)A_{qr}+S_{kl}n^1{}_{lt}(o)A_{ts}B_{spr}+(B_{spr}U_{iks}+U_{ijp}U_{jkr}+V_{ikpr})S_{kl}n^o{}_l(o)\}\cdot Z_p(o)Z_r(o).$$

After insertion of S, U, A, B "summed" over a period in the spin vector transformation formulas, we can find the periodical solution for $\mathbf{n}$.

Now let us rewrite $Y[\delta\mathbf{n}/\delta Y]$ from the polarization formula (the term which described deviation $\mathbf{n}$ on trajectory with only p_σ nonzero component in start point) in the form:

$$Y\,\frac{\delta\mathbf{n}}{\delta Y}=\frac{\mathbf{n}(s,\mathbf{Z}^{in}=(0,0,0,0,0,p_\sigma))-\mathbf{n}(s,0)}{p_\sigma}=$$
$$=S_{ij}n^1{}_{jr}(o)A_{r6}+U^1{}_{ik6}S_{kn}n^o{}_n(o)+O(p_\sigma)$$

using that in the magnet the $S_{zx}=S_{zx}=0$ and $S_{zz}=1$ we can write:

$$\delta\alpha_-=\int_o^s[n^o{}_z(o)-n^1{}_{zr}(o)A_{r6}+U^1{}_{zk6}S_{kn}n^o{}_n(o)]\;K^3ds$$

and similar formulae for $\delta\alpha_+$. Now we can analytically integrate $\delta\alpha_-$ and $\delta\alpha_+$ in the synchrotron magnet and express the adds into α_+ and α_- through $\mathbf{n}$ in entrance point.

The algorithm of degree of the equilibrium polarization calculation is the next:

First step: find transformation spin and matrix through period;

Second step: calculating periodically solving $\mathbf{n}_o$ and $\mathbf{n}_1(\mathbf{Z})$;

Third step: pulling $\mathbf{n}_o$ and $\mathbf{n}_1(\mathbf{Z})$ through structure elements and summing on the magnet of structure the adds of integrals α_+ and α_-: $\alpha_-=\Sigma\delta\alpha_-$, $\alpha_+=\Sigma\;\delta\alpha_+$.

SUPPLEMENT

The following values, which characterize the magnetic field, are introduced there:

$$K_{x,z}=\pm\,\frac{eH_{oz,ox}}{E_o},$$

$$g=\frac{e}{E_o}\cdot\frac{dH_z}{dx}=\frac{e}{E_o}\cdot\frac{dH_x}{dz},\qquad q=\frac{e}{2E_o}\cdot(\frac{dH_x}{dx}-\frac{dH_z}{dz}),$$

$$m_{x,z}=\frac{e}{2E_o}\cdot\frac{d^2H_{x,z}}{dxdz}=\frac{e}{2E_o}\cdot\frac{d^2H_{z,x}}{dx^2,dz^2}\;,$$

and the values of all quantities in the right sides are taken on the equilibrium orbit.

Thus, one obtains the final expressions for components of $\mathbf{W}$ (including the zero, first and second orders on x, z, p_σ and its derivatives $p_x=x'-\frac{1}{2}{}^e/_{E_o}H_{os}z$, $p_z=z'+\frac{1}{2}{}^e/_{E_o}H_{os}x$, p_σ):

$$W_x=-(B_{ox}+K_z)-B_{ox}\cdot(Y_oa+{}^a/_{2Y_o}+\tfrac{1}{2}/Y_o{}^2)+$$
$$+(\tfrac{1}{2}B'_{os}-q)\cdot(1+Y_oa)\cdot x+B_{os}a(Y_o-1)\cdot p_x+$$
$$+[\tfrac{1}{2}B_{os}{}^2a(Y_o-1)-(B_{ox}K_z+g)(1+Y_oa)]\cdot z+B_{ox}p_\sigma+$$
$$+\tfrac{1}{2}(gK_z-qK_x+m_z+B''_{ox})\cdot(Y_oa+1)\cdot x^2+B'_{ox}Y_oa\cdot x\cdot p_x-$$
$$-(gK_x+qK_z+m_x)\cdot(Y_oa+1)\cdot x\cdot z-(\tfrac{1}{2}B'_{os}-q)\cdot x\cdot p_\sigma+$$
$$+\tfrac{1}{2}B_{ox}(Y_oa-1)\cdot p_x{}^2+B'_{oz}Y_oa\;p_x\cdot z+B_{oz}Y_oa\cdot p_x\cdot p_z-$$
$$-(gK_z+\tfrac{1}{2}m_z)\cdot(Y_oa+1)\cdot z^2+(B_{ox}K_z+g)\cdot z\cdot p_\sigma-$$
$$-\tfrac{1}{2}B_{ox}\cdot(Y_oa+1)\cdot p_z{}^2-B_{ox}p_\sigma{}^2;$$

$$
\begin{aligned}
W_z = &- (B_{oz}-K_x)-B_{oz}\cdot(Y_o a+{}^{a}/2Y_o+\tfrac{1}{2}/Y_o{}^2)- \\
&- [\tfrac{1}{2}B_{os}{}^2 a\cdot(Y_o-1)+(B_{oz}K_x+g)\cdot(1+Y_o a)]\cdot x+ \\
&+ (\tfrac{1}{2}B'_{os}+q)(1+Y_o a)\cdot z+B_{os}a\cdot(Y_o-1)p_z+B_{oz}\cdot p_\sigma- \\
&- (gK_x+\tfrac{1}{2}m_x)\cdot(Y_o a+1)\cdot x^2 + \\
&+(qK_x-gK_z-m_z)\cdot(Y_o a+1)\cdot x\cdot z+B'_{ox}Y_o a\cdot x\cdot p_z + \\
&+(B_{oz}K_x+g)xp_\sigma-\tfrac{1}{2}B_{oz}(Y_o a+1)\cdot p_x{}^2+B_{ox}Y_o a\cdot p_x\cdot p_z+ \\
&+\tfrac{1}{2}(gK_x+qK_z+m_x+B''_{oz})(Y_o a+1)\cdot z^2+B'_{oz}Y_o a\cdot z\cdot p_z- \\
&-(\tfrac{1}{2}B'_{os}+q)\cdot z\cdot p_\sigma+ \tfrac{1}{2}B_{oz}(Y_o a-1)\cdot p_z{}^2-B_{oz}\cdot p_\sigma{}^2;
\end{aligned}
$$

$$
\begin{aligned}
W_s=&-B_{os}(1+a+{}^{1}/{}_{2}Y_o{}^2)-B'_{ox}(1+a)x-B'_{oz}(1+a)z+ \\
&+ B_{ox}a(Y_o-1)\cdot p_x+B_{oz}a(Y_o-1)\cdot p_z+B_{os}(1+a)\cdot p_\sigma- \\
&-\tfrac{1}{2}[\tfrac{1}{4}B_{os}{}^3\cdot(2Y_o a+1) + (q'-\tfrac{1}{2}B''_{os})]\cdot x^2+ \\
&+ (q-\tfrac{1}{2}\cdot B'_{os})\cdot Y_o a\cdot x\cdot p_x-g'\cdot x\cdot z+ \\
&+[gY_o a+B_{os}{}^2\cdot(Y_o a+\tfrac{1}{2})]\cdot x\cdot p_z+ \\
&+B'_{ox}x\cdot p_\sigma-\tfrac{1}{2}B_{os}\cdot(2Y_o a+1)\cdot p_x{}^2 - \\
&-[B_{os}{}^2\cdot(Y_o a+\tfrac{1}{2})-gY_o a]\cdot p_x\cdot z- \\
&-\tfrac{1}{2}[\tfrac{1}{4}B_{os}{}^3\cdot(2Y_o a+1)-(q'+\tfrac{1}{2}B''_{os})]\cdot z^2+ \\
&-(q+\tfrac{1}{2}B'_{os})Y_o a\cdot z\cdot p_z-\tfrac{1}{2}B_{os}(2Y_o a+1)\cdot p_z{}^2+ \\
&+B'_{oz}\cdot z\cdot p_\sigma-B_{os}p_\sigma{}^2.
\end{aligned}
$$

In this expressions y_o is the relativistic factor and $a=1.159\ldots\cdot 10^{-3}$ is the dimensionless part of the electron anomalous magnetic momentum.

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