ANALYTIC MODEL FOR QUADRUPOLE FRINGE-FIELD EFFECTS

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Abstract

The linear-field non-scaling FFAG lattices originally proposed for multi-GeV muon acceleration are now being modified for $\simeq 200$ MeV/u proton or carbon medical applications. The momentum range is large and the chromatic tune variation is significant. In the medical case, the issue of resonance crossing is acute owing to the lower rate of energy gain. Quadrupole magnets with non-normal entry/exit faces are considered as means to reduce the tune variation. Thus one is motivated to study the fringe-field effects of quadrupole magnets with rotated entry/exit faces. However, just as a bending (dipole) magnet fringe-field $B_f \propto re^{i\theta}$ is found to give a focusing effect, so we expect the fringe field of a quadrupole magnet, $B_f \propto r^2 e^{i2\theta}$, with rotated pole faces to give amplitude-dependent impulses reminiscent of a sextapole magnet. This note is a *precis* of two laboratory reports[11, 12].

INTRODUCTION

Our motivation is the design of non-scaling FFAGS, for proton therapy of cancer patients. In these machines, particle beams are accelerated over a wide range of momenta (100% or more) at constant magnetic field. The change in beam rigidity implies that betatron oscillation frequencies fall with momentum. It is anticipated that this effect can be mitigated by giving the higher momentum particles longer path lengths through the magnets, as is achieved by wedge-shaped elements.

Thus we consider fringe field effects for magnets with rotated pole faces; that is to say the entry and exit faces are not perpendicular to the reference trajectory through the magnet. Our derivation differs from those offered in standard texts [1, 2, 3, 4] in being entirely algebraic, which has the advantage that is may be readily adapted to an arbitrary multipole. Fringe-field effects for quadrupoles *without* pole-face rotations have a long history[5, 7, 8]. Fringe-field effects in dipoles have an even longer history[6] associated with their use as spectromter magnets.

QUADRUPOLE FRINGE FIELD EFFECTS

The basic configuration is that of a rectangular quadrupole magnet described in terms of a cartesian system $[x, y, z]$ with the z-axis colinear with the zero-field centre line of the magnet; and $x - z$ and $y - z$ being the horizontal and vertical symmetry planes. We consider the possibility of a rotated pole face, and take a system of coordinates $[q, y, p]$ for the fringe-field region, as in Figure 1. The situation is now as follows: we know the magnetic field components in terms of coordinates $[q, y, p]$, but we wish to find the equations of motion in terms of $[x, y, z]$ as an expansion about the reference trajectory in the drift.

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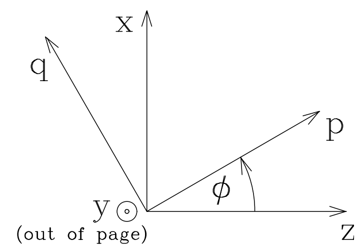

Figure 1: Orientation of coords $[x, y, z]$ versus $[q, y, p]$

Part of the difficulty of this problem derives from adopting the, in some sense, "wrong" coordinate system. It would be natural to develop the fringe-field motion in $[q, y, p]$, but we insist on $[x, y, z]$. Typically we do this to avoid rotations of the momenta (between coordinate systems) and because the fringe is typically much shorter (and weaker) than the body field, so that the motion is dominated by the body field and its coordinate system. However, some authors[9, 10] do introduce the rotations, albeit for thin-element fringe fields.

Transformation of field and coordinates The two coordinate systems are related by $[q, y, p] = \mathbf{T}[x, y, z]$ and $[z, y, z] = \mathbf{T}^{-1}[q, y, p]$, where $\mathbf{T}^{-1}$ denotes the inverse of the matrix

$$\mathbf{T} = \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix} .$$

Fringe field with rotated pole face

Contrary to the case of the dipole, the quadrupole magnetic field is not rotation symmetric about the y-axis; and when the pole face is not cut perpendicular to the magnet principal axis, z, it is not immediately clear how the exterior fringe field relates to the interior body field. However, some "asymptotic" properties are evident. Sufficiently deep within the interior, the body field is essentially two-dimensional; and sufficiently far from the pole face, the fringe field is dominated by the field source (i.e. the four pole pieces) and the "longitudinal" component B_p is directed perpendicular and away from the rotated pole face.

Transition region The issue then, is how large and how significant is the transition region between these two "asymptotic" zones where the field shapes are relatively simple. To answer that question, we have performed some 3D-modeling (Using Vector Fields Opera-3D) of a

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quadrupole magnet with entry/exit faces rotated about the y-axis. In our model, Fig.2, we took a magnet 10 units in length along its centreline, and having the radius to the hyperbolic pole tips of 3 units. The pole faces were then cut at $10°$ to the perpendicular. Figs. 3,4 show magnetic field components and contours in the transition and fringe regions. For this model, the 2-D interior body field is established within 1/2 unit of the pole face; and the exterior fringe field approaches the simple 3-D "asymptotic" form within 1 unit distance of the poleface. The finge field falls almost to zero within 5 units distance of the poleface. (This is typical of a quadrupole: the longitudinal extent of the fringe is roughly twice the radius to poletip.) The conclusion is that eliminating the details of the transition region from our analytical model will little compromise our results because the "asymptotic" forms are established within a short distance either side of the pole face.

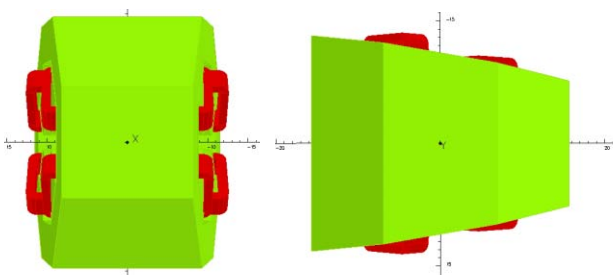

Figure 2: X (left) and Y (right) elevations of quadrupole.

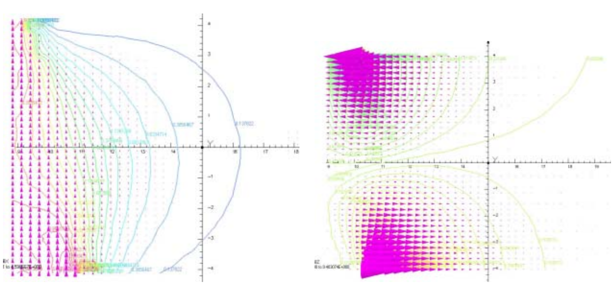

Figure 3: B_x (left) and B_z (right) field components.

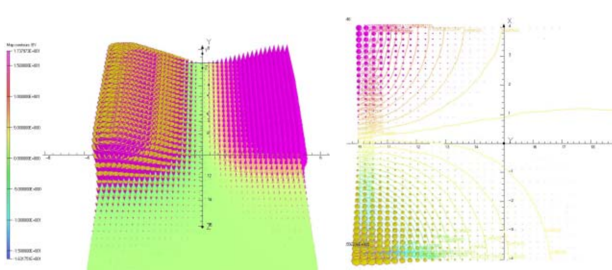

Figure 4: B_y field components and contours.

Fringe field - linear in transverse coords

Let B_1 (tesla/metre) be the quadrupole gradient. Let $\mathbf{B}_{xyz} = [B_x, B_y, B_z]$ and $\mathbf{B}_{qyp} = [B_q, B_y, B_p]$. In the interior region, the potential and body field are:

$$\Phi = -B_1 xy \ \text{ and } \ \mathbf{B}_{xyz} = -\nabla\Phi = [B_1 y, B_1 x, 0] \ .$$

Let $W(bp)$ model the fringe-field fall off, as above. In the exterior region, the potential and fringe field are:

$$\begin{aligned} \Phi_f &= -B_{1f}\, qy\, W \ \text{ and} \\ \mathbf{B}_{qyp} &= -\nabla\Phi_f = B_{1f}[yW, qW, qybW'] \ . \end{aligned}$$

Notice how the longitudinal component B_p has the same form in the $q-y$ plane as the 2-D potential from which it is derived; the angular symmetry is quadrupolar (from the pole face geometry), but the radial ($r = \sqrt{q^2+y^2}$) dependence is quadratic.

For the entrance fringe field, the field is increasing and so $W' > 0$ which implies the B_p component is reversed compared with the exit fringe for which $W' < 0$.

Transformation of field and coordinates The next step is to investigate continuity of field and potential in the boundary plane $p = 0$. To do that, one must transform the fringe field components in $[q, y, p]$ into components directed along $[x, y, z]$. The components are $\mathbf{B}_{xyz} = \mathbf{T}^{-1}\mathbf{B}_{qyp}$ or, explicitly,

$$\frac{\mathbf{B}_{xyz}}{B_1} = \begin{bmatrix} y\cos\phi\, W(bp) \ + bqy\sin\phi\, W'(bp) \\ qW(bp) \\ -y\sin\phi\, W(bp) \ + bqy\cos\phi\, W'(bp) \end{bmatrix} .$$

At the point $[q, y, 0]$, we find coords $[x, y, z] = [q\cos\phi, y, -q\sin\phi]$. Hence $\Phi = -B_1 qy\cos\phi$ while $\Phi_f = -B_{1f}qy$. The interior field has components $[B_x, B_y, B_z] = B_1[y, q\cos\phi, 0]$ while the exterior field has components $[B_x, B_y, B_z] = B_{1f}[y\cos\phi, q, -y\sin\phi]$. Clearly these functions are not continuous across the boundary unless $\phi = 0$; and this was to be expected because we have eliminated the transition region and the field rotations that occur there. We adopt a compromise, set $B_{1f} = B_1$ and incur a scale error of order $\cos\phi$. The next step is to re-write $[q, y, p]$ in terms of $[x, y, z]$ using $[q, y, p] = \mathbf{T}[x, y, z]$, leading to $p = (z\cos\phi + x\sin\phi)$ and $q = (x\cos\phi - z\sin\phi)$.

Expansion about reference trajectory The next step is to make an expansion of this field about the reference trajectory $[x, y, z] = [0, 0, z]$. We pursue the course *not* to expand $W(bZ)$, etc, in powers of Z. But what to do with the mixed powers of x and z arising from q^n? We shall accept all terms $x^i y^j$ (irrespective of z^k) that satisfy $i + j = n$ for a calculation of order n. To first order, with n=1, the result is

$$\frac{\mathbf{B}_{xyz}}{B_1} = \begin{bmatrix} y\cos\phi \\ (x\cos\phi - Z\tan\phi) \\ -y\sin\phi \end{bmatrix} W - \begin{bmatrix} y\sin\phi\tan\phi \\ x\sin\phi\tan\phi \\ y\sin\phi \end{bmatrix} bZW' .$$

The next step is to find the forces $\mathbf{F} = e\,\mathbf{U}\wedge\mathbf{B}_f$. $\mathbf{U}$ is the velocity vector. Let $\mathbf{F}_{xyz} = [F_x, F_y, F_z]$. We neglect terms in $\dot{x}$ and $\dot{y}$ as small compared with v_s, and obtain the components

$$\frac{\mathbf{F}_{xyz}}{B_1 e v_s} = \begin{bmatrix} -x[\cos\phi\, W - bZ\sin\phi\tan\phi\, W'] + Z\tan\phi\, W \\ +y[\cos\phi\, W - bZ\sin\phi\tan\phi\, W'] \\ 0 \end{bmatrix} .$$

Clearly we have focusing/defocusing terms in x and y (respectively). Under the assumption $W \geq 0$, the body field is horizontally focusing. In addition there is a dipole term in F_x; this should come as no surprise, because our geometry implies the particle travels off-axis in the exterior quadrupolar fringe field. F_x bends according to which side the z axis lies compared with p. In the case $\phi > 0$, the bend is oppositely directed to the horizontal focusing.

Deflections We find thin-lens type approximate expressions for the deflexions produced by these fields under the condition that the fringe is sufficiently narrow that there is no change in the coordinate values within the fringe. Let us introduce the parameter

$$\rho \equiv (\gamma m_0 v_s)/[e(B_1 L)] \,, \tag{1}$$

which is the analogue of the dipole bending radius only with $B_1 \times L$ replacing B_0. The change in divergences are:

$$\begin{aligned} \Delta y' &= +(y/\rho)\bar{W}\sec^2\phi & (2)\\ \Delta x' &= -(x/\rho)\bar{W}\sec^2\phi \,+ (\bar{L}/\rho)\sec\phi\tan\phi \,. & (3)\end{aligned}$$

Here $\bar{W}$ is the average value of W; and $\bar{L} = \int_0^L ZW(bZ)dZ$. For example, if $W(s) = \cos^2(s)$ then $\bar{W} = 1/2$ and $\bar{L} \approx L/6$.

Nonlinear fringe field

To find the second order terms in the fields and forces, we have first to find the higher order correction to the potential function in the region exterior to the quadrupole. We appeal to the Fourier-Bessel expansion for guidance on the correct form. The relevant term for a quadrupole is $I_2(kr)e^{ikz}e^{j2\theta}$ and the radial dependence is $I_2(kr) \approx (kr/2)^2/2 + (kr)^4/6 + \ldots$. Hence the trial potential function for the fringe

$$\Phi_f = -G(p)qy \,+ H(p)qy(q^2 + y^2) \,.$$

Substitution into Laplace's equation gives the approximate condition $H(p) = G''/12$ and the error in the Laplacian is reduced by the factor $\varepsilon \approx (q^2 + y^2)b^2/12$. We do not have to re-match the potential on the boundary plane $p = 0$, because the higher order terms are small provided that $\varepsilon \ll 1$. Starting from the fringe field potential

$$\Phi_f = B_1 qy[-W(bp) \,+ (q^2 + y^2)(b^2/12)W''(bp)] \,,$$

we repeat the previous steps of field transformation and coordinate substitution, and expand about the reference trajectory $[x, y, z] = [0, 0, z]$. The *additional* nonlinear field is

$$\frac{\Delta \mathbf{B}_{xyz}}{B_1} = \begin{bmatrix} xy\sin 2\phi \\ x^2\cos\phi\sin\phi \\ xy\cos 2\phi \end{bmatrix} b\,W' + \frac{1}{4}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} b^2 ZW''.$$

$$\begin{aligned} c_1 &= -y(x - 3x\cos 2\phi + Z\sin\phi)\tan\phi \\ c_2 &= x^2\cos\phi\sin\phi \\ &+ \tan\phi[y^2 - x\sin\phi(Z + 2\sin\phi) + (Z^2/3)\tan^2\phi] \\ c_3 &= y(Z\sin\phi - 6x\cos^2\phi)\tan^2\phi \,.\end{aligned}$$

Forces and deflections The next step is to find the additional forces: $\Delta F_x = -ev_s\Delta B_y$, $\Delta F_y = +ev_s\Delta B_x$ and $\Delta F_z = 0$. By integration of these forces, we find the *additional* deflections:

$$\begin{aligned} \Delta y' &= -[xy/(L\rho)][(1/2) + \sec^2\phi]\sin\phi & (4)\\ \Delta x' &= +[x^2/(2L\rho)][(1/2) + \sec^2\phi]\sin\phi & (5)\\ &- [y^2/(4L\rho)]\sec\phi\tan\phi \\ &+ [x/(2\rho)]\bar{W}\tan^2\phi \,- [\bar{L}/(2\rho)]\sec\phi\tan^3\phi \,.\end{aligned}$$

These higher order terms are of order (x/L) smaller than the quadrupole fringe focusing/defocusing terms; and become important when the amplitude of the horizontal oscillation is comparable with the length of the fringe field.

If we consider that there are large angles, then there are further terms appearing in the forces because we can no longer omit $\dot{x}, \dot{y}$.

$$\frac{\Delta \mathbf{F}_{xyz}}{eB_1 v_s} = \begin{bmatrix} -yy'\sin\phi \\ +yx'\sin\phi \\ (xx' - yy')\cos\phi - x'Z\tan\phi \end{bmatrix} W(bZ) + \begin{bmatrix} +y' \\ -x' \\ 0 \end{bmatrix} y\frac{b^2Z^2}{4}W''(bZ)\sin\phi\tan^2\phi \,.$$

The resulting additional deflections, compared with equations (2, 3) are:

$$\begin{aligned} \Delta x' &= -y'(y/\rho)\bar{W}\tan\phi[1 - (\tan^2\phi)/2] & (6)\\ \Delta y' &= +x'(y/\rho)\bar{W}\tan\phi[1 - (\tan^2\phi)/2] & (7)\\ \Delta z' &= (xx' - yy')\bar{W}/\rho \,- (\bar{L}/\rho)x'\sec\phi\tan\phi \,. & (8)\end{aligned}$$

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