IMAGE EFFECTS ON THE TRANSPORT OF INTENSE BEAMS*

R. Pakter†, Y. Levin, F. B. Rizzato
Insituto de Física, Universidade Federal do Rio Grande do Sul, Brazil

Abstract

The effects of a conducting pipe on the equilibrium of intense nonaxisymmetric continuous beams are investigated. First, we analyze the image effects of a cylindrical conducting pipe on a beam with elliptical symmetry. It is derived an exact expression for the self-field potential of the beam inside the pipe without using any sort of multipole expansion. By means of a variational method we search for equilibrium solutions for such intense beams. Finally, we prove that despite the nonlinear forces imposed by the image charges of an arbitrary shape conducting pipe, equilibrium intense beams preserve density homogeneity and their free-space transverse sizes.

INTRODUCTION

In many applications where intense charged particle beams are employed, it is desirable to have nonaxisymmetric beam distributions [1, 2]. The focusing field configuration used to transport nonaxisymmetric beams is based on anisotropic periodic focusing fields that generate stronger effective focusing in one transverse direction. Zhou et.al. [2] demonstrated the existance of a class of equilibrium solutions for the transport of intense nonaxisymmetric beams with variable aspect ratios through such periodic magnetic focusing fields in the free space. In these equilibria, the beam is uniformly distributed along an ellipsis whose angle and semi-axis radii present just some small-amplitude fast oscillations around stationary average values. An issue that comes into mind, therefore, is how such equilibrium beams would be affected by the presence of the walls of a vacuum chamber. While round beams are not disturbed by the presence of a coaxial round conducting pipe, because the pipe is naturally an equipotential, elliptical beams may be heavily affected by them. In particular, one may expect that the charges of opposite sign induced at the wall will strongly attract beam particles, causing deviations in the beam equilibrium shape and distribution homogeneity.

MODEL

We consider an intense, unbunched beam propagating with average axial velocity v_z through a magnetic focusing channel and contained in a conducting pipe, both aligned with the z axis. The focusing force is assumed to be linear and anisotropic along the transverse directions. In the smooth-beam approximation, where the fast oscillations due to the periodic nature of the focusing field are averaged out, the dynamics of a beam charge is dictated by [1, 3]

$$\mathbf{r}'' + \nabla_\perp U_B + \nabla_\perp \psi = 0, \tag{1}$$

where $\mathbf{r} = x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y$, $r = (x^2 + y^2)^{1/2}$ is the radial distance from the z axis, the prime denotes derivative with respect to z, $\nabla_\perp \equiv (\partial/\partial x)\hat{\mathbf{e}}_x + (\partial/\partial y)\hat{\mathbf{e}}_y$, $U_B = k_x x^2/2 + k_y y^2/2$ is the effective confining potential due to the external field, $k_i = \xi_i^2 q^2 \overline{B(z)^2}/2\gamma_b^2\beta_b^2 m^2 c^4$, $i = x, y$, $B(z)$ is the magnetic field along the z axis, the bar represents average over one focusing period, ξ_i are form factors that satisfy $\xi_x + \xi_y = 1$, $\beta_b = v_z/c$, $\gamma_b = (1 - \beta_b^2)^{-1/2}$, q and m are the mass and charge of the beam particles, and c is the speed of light in *vacuo*. In Eq. (1), ψ is a normalized potential that incorporates both self-electric and self-magnetic fields and is also affected by the presence of the conducting pipe. It is related to the self-scalar and self-vector potentials by $\phi^s = \beta_b^{-1} A_z^s = \gamma_b^3 m \beta_b^2 c^2 \psi(\mathbf{r}, s)/q$ and solves the Poisson Equation

$$\nabla_\perp^2 \psi = -(2\pi K/N_b)\, n_b(\mathbf{r}, z), \tag{2}$$

subjected to boundary condition $\psi =$ const. at the conducting pipe, where $n_b(\mathbf{r}, z)$ is the beam density profile, $N_b =$ const. is the number of particles per unit axial length, and $K = 2q^2 N_b/\gamma_b^3 \beta_b^2 mc^2$ is the so-called beam perveance that can be seen as a measure of the total two-dimensional beam charge, which is precisely $K/2$.

SELF-FIELD POTENTIAL WITH CONDUCTOR

To start, let us specialize to the case of an elliptical beam propagating inside a cylindrical conducting pipe of radius r_w and look for an exact solution for the self-field potential. In order to investigate how the image charges may affect the beam distribution, we take into consideration in our derivations a non-uniform distribution. Namely, we assume a parabolic density profile of the form

$$n(x, y) = \frac{N_b}{\pi ab}\left[1 + \chi - 2\chi\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)\right] \tag{3}$$

inside the beam core, $(x/a)^2 + (y/b)^2 \leq 1$, where χ is the inhomogeneity parameter $-1 \leq \chi \leq 1$, and a and b are the ellipsis semi-axis radii. In the absence of the conductor, the beam is in free space and the solution to the Poisson

* Work supported by CNPq and FAPERGS, Brazil, and by the US AFOSR, grant n. FA9550-06-1-0345.
† pakter@if.ufrgs.br.

Equation is known. Inside the beam, the self-field is given by [3]

$$\psi_{in}^{free}(\mathbf{r}) = -\frac{K_b}{2}\int_0^\infty \frac{(1+\chi)T - \chi T^2}{[(a^2+s)(b^2+s)]^{1/2}}\,ds, \quad (4)$$

where $T \equiv x^2/(a^2+s)+y^2/(b^2+s)$. This integral can be calculated, leading to an analytical expression for ψ_{in}^{free}; for the sake of conciseness we do not present it here. Although there exists an integral form equivalent to Eq. (4) for the self-field outside of the beam, ψ_{out}^{free}, it can only be solved numerically. Nevertheless, one can obtain an analytic expression by noting that in the far-field region ($r \to \infty$) the self-field has to scale like that of a single particle at the origin, $\psi_{out}^{free} \approx \log r$, and by imposing continuity of the electric field at the beam boundary [4]. Performing these calculations and taking advantage of the properties of complex variable functions, we can suitably write

$$\psi_{out}^{free}(\mathbf{r}) = Re\left[\frac{3+\chi}{6}\mathcal{H} + \frac{\chi}{12}\mathcal{H}^2 - \arccos\left(\frac{\zeta}{c}\right)\right], \quad (5)$$

where $\zeta = x + iy$, $i = \sqrt{-1}$, $\mathcal{H} = 1 - 2(\zeta/c)^2[1 - \sqrt{1-(c/\zeta)^2}]$, and $c = \sqrt{a^2-b^2}$. With a complete free-space solution in hands, we now proceed to develop a general method to include the effects of a cylindrical pipe. For that, let us consider an arbitraty beam distribution of total charge $K/2$ that is all contained in the region $r < r_w$. Thus, the free-space self-field satisfies Laplace equation $\nabla_\perp^2\psi^{free} = 0$ for $r > r_w$. Considering cylindrical coordinates, one can then verify that the function $\psi^{free}(r_w^2/r,\theta)$ solves Laplace equation for $0 < r < r_w$ and presents a singularity at $r = 0$ that corresponds to a point of charge $K/2$ sited there. In other words, for *each* source charge of $\psi^{free}(r,\theta)$ located at $r < r_w$, it can be shown that there corresponds two source charges of $\psi^{free}(r_w^2/r,\theta)$: one *image* charge placed at $r > r_w$ and one *spurious* charge at the origin. Because all the spurious charge concentrate at $r = 0$, we can easily subtract their contribution. We therefore construct the self-field

$$\psi(\mathbf{r}) = \psi^{free}(r,\theta) - \psi^{free}(r_w^2/r,\theta) + K\log(r/r_w) \quad (6)$$

which solves Poisson equation with the proper source charges for $r < r_w$ and the proper boundary condition at $r = r_w$, namely, $\psi(r = r_w,\theta) = 0$; the last term is the one responsible for compensating the spurious charges. It is worth noting that although we are only interested in the self-field $\psi(\mathbf{r})$ inside the pipe, we ought to know $\psi^{free}(\mathbf{r})$ everywhere in order to include wall effects. Substituting Eqs. (4) and (5) in Eq. (6) we obtain an exact analytic expression for the self-field of an elliptical beam inside a round pipe.

VARIATIONAL PRINCIPLE

For a given set of transport channel parameters k_x, k_y, and r_w we numerically compute the total beam energy per particle

$$\mathcal{E}_T = \frac{1}{N_b}\int\left[\frac{\psi(\mathbf{r})}{2} + U_B(\mathbf{r})\right]n(\mathbf{r})\,d^2r, \quad (7)$$

as a function of the beam parameters a, b and χ. By minimazing $\mathcal{E}_T(a,b,\chi)$ we can then find an approximation to the beam equilibrium solution. For $r_w \to \infty$ we can analytically solve this problem to find that the equilibrium correponds to a uniform beam ($\chi = 0$) with $a = a_0 \equiv \sqrt{2Kk_y/[k_x(k_x+k_y)]}$ and $b = b_0 \equiv \sqrt{2Kk_x/[k_y(k_x+k_y)]}$. As it seems more natural, instead of using k_x and k_y, we use the free-space equilibrium radii a_0 and b_0 to characterize the focusing field intensity. In Fig. 1 we show the results obtained for $r_w/a_0 = 1.2$ and varying values of b_0/a_0. For later comparison with self-consitent simulations, we present in panel (a) the equilibrium *effective* semi-axis $a_{rms} \equiv 2\langle x^2\rangle^{1/2} = a(1-\chi/3)^{1/2}$ and $b_{rms} \equiv 2\langle y^2\rangle^{1/2} = b(1-\chi/3)^{1/2}$, where $\langle\cdots\rangle$ stands for the average over beam distribution. The figure confirms that for nearly axisymmetric beams with $b_0/a_0 \approx 1$, wall effects are negligible such that a_{rms}/a_0 and b_{rms}/b_0 are close to unity. As the focusing channel becomes anisotropic with $b_0/a_0 < 1$, wall effects become noticeable and always act in the sense of further intensifying beam anisotropy by increasing its size along x and decreasing along y. The figure also reveals that the dependence of the equilibrium beam sizes on the focusing field anisotropy b_0/a_0 is non monotonic, being more pronounced for aspect ratios close to $b_0/a_0 = 0.5$. This feature was tested and verified for different wall radius r_w as well. Self-consistent simulations were performed to verify the results from the variational principle. In the simulations, a large number $N = 10000$ of macroparticles evolve according to Eq. (1), where the self-potential is calculated via Green's Function method. The particles are lauched in an arbitrary distribution and attain the equilibrium state by introducing a slow dumping in their dynamics. The simulation results obtained for a_{rms} and b_{rms} are represented by the symbols in Fig. 1(a), showing a very good agreement with the results from the variational principle. In Fig. 1(b), it is presented the results obtained with the variational method for the inhomogeneity parameter. It shows that χ is also a non-monotonic function of b_0/a_0. More importantly, though, is the fact that χ always presents modest values of a few percent. This not only implyes that the disturbance on the distribution homogeneity due to the wall is small, but also indicates that because χ is always positive the beam density that minimizes the energy presents a bell-shaped distribution with slightly lower densities at the beam boundary. This contrasts with the *natural* idea that the charge of opposite sign induced at the conductor pipe would attract more strongly the beam charges that are closer to the wall, increasing beam density there. Another intriguing property of the equilibrium calculated with the aid of the variational principle is that the effetive area occupied by the beam in the presence of the wall is exactly the same as that of the free beam, i.e., $a_{rms}b_{rms} = a_0b_0$ for all aspect

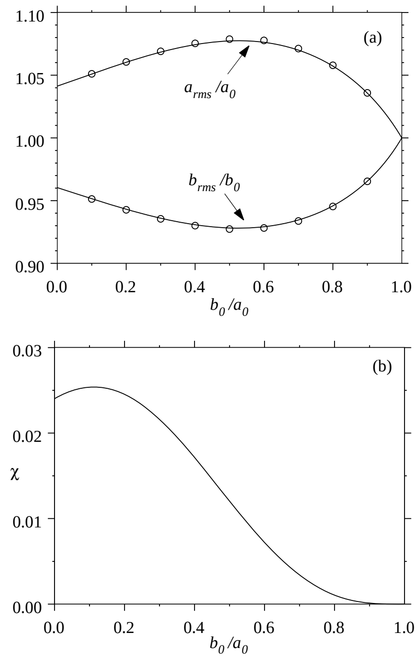

Figure 1: Equilibrium beam parameters as a function of the aspect ratio b_0/a_0 for $r_w/a_0 = 1.2$. The curves correspond to results from the variational principle, whereas the symbols in panel (a) to those obtained from self-consistent simulations.

ratios [see Fig. 1(a)]. In principle, this can be a mere coincidence. However, such features in variational calculations may also indicate the existence of some sort of hidden symmetry. With that in mind, we were able to determine an important property of the equilibrium beam density in the presence of a conducting pipe as discussed below.

EQUILIBRIUM BEAM DENSITY

Let us consider an equilibrium beam in the absence of walls. Because each particle is in equilibrium we obtain from Eq. (1) that $\nabla_\perp U_B + \nabla_\perp \psi = 0$. If we picture the beam as a continuous distribution of charges, this equation imposes a condition that has to be satisfied by the force vector field inside the equilibrium beam distribution. We now suppose that a conductor of an arbitrary shape initially at the infinit is adibatically approaching the beam. The distribution deforms due to the wall presence, but because the wall motion is adiabatic, the equilibrium condition imposed to the force vector field holds. Operating with $\nabla_\perp$ on the equilibrium force equation we can write

$$\nabla_\perp^2 U_B(\mathbf{r}) - (2\pi K/N_b)\, n_b(\mathbf{r}, r_w) = 0, \quad (8)$$

where r_w, here, stands for typical distance from the focusing channel axis to the condutor and use has been made of the Poisson equation (2). Note that while n_b is a function of r_w because the beam changes as the wall approaches, $U_B(\mathbf{r})$ only depends on the external field, being r_w independent. As the wall suffers a small displacement δr_w with $r_w \to r_w + \delta r_w$, beam particles positions slightly change according to $\mathbf{r} \to \mathbf{r} + \delta\mathbf{r}$. Using that in Eq. (8) and expanding to linear order in δr_w and $\delta \mathbf{r}$ we obtain

$$\left[\frac{\partial}{\partial r_w} + \mathbf{v}.\nabla_\perp\right] n_b(\mathbf{r}, r_w) = \frac{N_b}{2\pi K}\mathbf{v}.\nabla_\perp\left(\nabla_\perp^2 U_B\right), \quad (9)$$

where $\mathbf{v} \equiv \delta\mathbf{r}/\delta r_w$. As long as U_B is a quadratic funtion of $\mathbf{r}$ – i.e., the focusing force is linear – the right-hand-side of Eq. (8) vanishes and the total (convective) derivative of $n_b(\mathbf{r})$ with respect to r_w variations is zero. This means that as the wall approaches from the infinity, particles move and change beam shape but always preserve the constant value of n_b inside the beam. In other words, equilibrium beam boundary format may change with the presence of an arbitraby conducting pipe, but the density homogeneity and the total area occupied by the beam are conserved. Note that this derivations are valid irrespective to the intrinsic nonlinear nature of the the image charge forces. Such condition clearly imposes a severe constrain on how nearby conductors affect beam equilibria.

CONCLUSIONS

To conlcude, we have investigated the effects of a conducting pipe in the equilibrium of intense nonaxisymmetric beams. We first analyzed the image effects of a cylindrical conducting pipe on a continuous beam with elliptical symmetry and derived an exact expression for the self-field potential of the beam inside the pipe. By means of a variational method, we found that wall effects increase the equilibrium beam anisotropy and have a greater impact on beams with aspect ratios $b_0/a_0 \approx 0.5$. Finally, we proved that despite the nonlinear forces imposed by the image charges of an arbitrary shape conductor, intense beams preserve a uniform density in the equilibrium as long as the focusing forces are linear. Not only that, the total area occupied by the beam is the same as it would be in the absence of the conductor. This severely limits the effect of nearby conductors on the beam equilibria and is anticipated to have practical relevance in the design of intense beam transport channels.

REFERENCES

[1] J. H. Booske, B. D. McVey, and T. M. Antonsen, Jr., J. Appl. Phys. **73**, 4140 (1993).

[2] J. Zhou, R. Bhatt, and C. Chen, Phys. Rev. ST Accel. Beams **9**, 034401 (2006).

[3] R.C. Davidson and H. Qin, *Physics of Intense Charged Particle Beams in High Energy Accelerators* (World Scientific, Singapore, 2001).

[4] S. Koscielniak, in *Proceedings of the 2001 Particle Accelerator Conference*, 2970 (2001).

