ANALYTICAL CALCULATION OF THREE-DIMENSIONAL MAGNETIC FIELDS FOR INSERTION DEVICES

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Abstract

To analyze how insertion devices affect on an electron beam dynamics, it is necessary to know theirs three-dimensional distribution of the magnetic field. This is of particular value for non-standard wiggler designs, such as with trapezoidal or staggered magnetic blocks, or with alternate pole canting. It is generally shown in this paper that by the knowing of two-dimensional distribution of magnetic field at the median plane of the insertion device, it is possible to calculate the three-dimensional distribution of this field over all space. The analytical expressions for the magnetic field components are derived for the general case, both for planar and non-planar insertion devices. The obtained magnetic field satisfies the Maxwell equations.

1 INTRODUCTION

Nowadays the most accurate information about magnetic fields of insertion devices comes from a numerical or analytical calculation. At the same time, it is necessary in some occasions to use more realistic description of the magnetic field. Electron trajectory and spin motion can be numerically calculated using field map along beam trajectory. But the map usually has some errors and thus the magnetic field, either simulated or measured, not always satisfy Maxwell equations. To overcome this problem, usually two methods of improving of the magnetic field data are used. The first one implies some small modification of data to make them satisfy Maxwell equations. The second method implies the calculation or measuring one or two components of the field with the following recovering of other components of magnetic field. Second method sounds much more physically (see [1]), and this is a method whereby magnetic fields of helical coils were calculated [2]. This method was also applied for periodic magnetic systems [3] and for helical wigglers [4]. Within the context of this approach, the following problem should be solved: which set of magnetic field data is necessary and sufficient for unique recovering of magnetic field over all space. It has been theoretically proved in this paper that if all three components of magnetic field are given at the median plane of insertion device, its magnetic field can be recovered uniquely over all space. This result is obtained for a general case, both planar and non-planar, periodical or non-periodical insertion devices.

2 MAGNETIC FIELD CALCULATION

Let us consider the right-hand Cartesian system of coordinates with the x and y axis directed horizontally and z axis directed upwards. Magnetic field

$$\vec{H}\left(x,y,z\right)=\left\{H_x\left(x,y,z\right),H_y\left(x,y,z\right),H_z\left(x,y,z\right)\right\} \tag{1}$$

satisfies Maxwell equations:

$$\begin{aligned} div\vec{H}\left(x,y,z\right)&=0 \\ rot\vec{H}\left(x,y,z\right)&=0 \end{aligned} \tag{2}$$

It follows from equations (2) that $\vec{H}\left(x,y,z\right)$ satisfies

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\vec{H}\left(x,y,z\right)=0 \tag{3}$$

The plane $x0y$ is assumed to be free of magnetic material and every of three components of magnetic field are specified at the plane $x0y$:

$$\begin{aligned} &H_x\left(x,y,z=0\right) \\ &H_y\left(x,y,z=0\right) \\ &H_z\left(x,y,z=0\right) \end{aligned} \tag{4}$$

and the following relation is fulfilled:

$$\frac{\partial}{\partial x}H_y\left(x,y,z=0\right)=\frac{\partial}{\partial y}H_x\left(x,y,z=0\right) \tag{5}$$

Let us find now the components of magnetic field $H_{x,y,z}\left(x,y,z\right)$ at $z\neq 0$. $H_z\left(x,y,z\right)$ may be expressed through the Maclaurin series:

$$H_z\left(x,y,z\right)=\sum_{n=0}^{\infty}\frac{z^n}{n!}\left(\frac{\partial^n}{\partial z^n}H_z\left(x,y,z\right)\right)_{z=0} \tag{6}$$

It is easy to derive from (3) that:

$$\begin{aligned} &\left(\frac{\partial^{2k}}{\partial z^{2k}}H_z\left(x,y,z\right)\right)_{z=0} \\ &=(-1)^k\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^k H_z\left(x,y,z=0\right) \\ &\qquad k=0,1,2,3,\cdots \end{aligned} \tag{7}$$

and hence:

$$\begin{aligned} &\left(\frac{\partial^{2k+1}}{\partial z^{2k+1}}H_z\left(x,y,z\right)\right)_{z=0} \\ &=(-1)^k\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^k\left(\frac{\partial H_z\left(x,y,z\right)}{\partial z}\right)_{z=0} \\ &\qquad k=0,1,2,3,\cdots \end{aligned} \tag{8}$$

The expression (6) can be written as:

$$\begin{aligned} H_z\left(x,y,z\right)&=\sum_{k=0}^{\infty}\frac{z^{2k}}{(2k)!}\left(\frac{\partial^{2k}}{\partial z^{2k}}H_z\left(x,y,z\right)\right)_{z=0} \\ &+\sum_{k=0}^{\infty}\frac{z^{2k+1}}{(2k+1)!}\left(\frac{\partial^{2k+1}}{\partial z^{2k+1}}H_z\left(x,y,z\right)\right)_{z=0} \end{aligned} \tag{9}$$

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By the substituting (7) and (8) into (9), we get:

$$H_z(x,y,z)=\sum_{k=0}^{\infty}\frac{z^{2k}}{(2k)!}(-1)^k\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^k \times H_z(x,y,z=0)+\sum_{k=0}^{\infty}\frac{z^{2k+1}}{(2k+1)!} \times(-1)^k\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^k\left(\frac{\partial H_z(x,y,z)}{\partial z}\right)_{z=0} \tag{10}$$

One can find from $div\vec{H}(x,y,z)=0$ that:

$$\left.\frac{\partial H_z(x,y,z)}{\partial z}\right|_{z=0} = -\left(\frac{\partial H_x(x,y,z=0)}{\partial x}+\frac{\partial H_y(x,y,z=0)}{\partial y}\right) \tag{11}$$

where right side is specified by the initial conditions (4). It immediately follows that:

$$H_z(x,y,z)=\sum_{k=0}^{\infty}\frac{z^{2k}}{(2k)!}(-1)^k\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^k \times H_z(x,y,z=0) -\sum_{k=0}^{\infty}\frac{z^{2k+1}}{(2k+1)!}(-1)^k\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^k \times\left(\frac{\partial H_x(x,y,z=0)}{\partial x}+\frac{\partial H_y(x,y,z=0)}{\partial y}\right) \tag{12}$$

Since the equations (2) include the following relations:

$$\frac{\partial}{\partial z}H_y(x,y,z)=\frac{\partial}{\partial y}H_z(x,y,z) \qquad \frac{\partial}{\partial z}H_x(x,y,z)=\frac{\partial}{\partial x}H_z(x,y,z) \tag{13}$$

we can derive the following expressions for H_y and H_x over all space:

$$H_y(x,y,z) = H_y(x,y,z=0)+\int_0^z\frac{\partial H_y(x,y,z')}{\partial z'}dz' = H_y(x,y,z=0)+\int_0^z\frac{\partial H_z(x,y,z')}{\partial y}dz' \tag{14}$$

and similarly:

$$H_x(x,y,z) = H_x(x,y,z=0)+\int_0^z\frac{\partial H_x(x,y,z')}{\partial z'}dz' = H_y(x,y,z=0)+\int_0^z\frac{\partial H_z(x,y,z')}{\partial x}dz' \tag{15}$$

Now we can state the following result. Given the three components of magnetic field in the median plane $x0y$ (see (4)), which satisfy the relation (5), one can find the magnetic field distribution over all space by the use of Eqs. (12), (14) and (15)). It is easy to find that the resulting magnetic field satisfies Maxwell equations (2).

This result may be explained in the following manner. Maxwell equations for static magnetic field consist of four equations. These equations forms the linear system of equations for derivatives of components of the magnetic field. So we have four equations for the nine unknown quantities. To solve uniquely this system, we have to define five of nine quantities. As it was mentioned above, we specify the distribution of magnetic field in the median plane $x0y$ (see (4)). It means, that the following quantities are specified: $\frac{\partial H_x}{\partial x}$, $\frac{\partial H_x}{\partial y}$, $\frac{\partial H_y}{\partial x}$, $\frac{\partial H_y}{\partial y}$, $\frac{\partial H_z}{\partial x}$ and $\frac{\partial H_z}{\partial y}$ (for example, we can calculate them numerically). Total number of just listed quantities is six. But magnetic field has to satisfy Eq. (5), and as a consequence two of these six components are dependent. So we have five independent quantities as initial conditions, and we can calculate from Maxwell equations three others quantities, namely: ($\frac{\partial H_x}{\partial z}$, $\frac{\partial H_y}{\partial z}$ and $\frac{\partial H_z}{\partial z}$).

3 SYMMETRIC MAGNETIC SYSTEM

Let us assume that the magnetic system is symmetric with respect to the plane $x0y$. By this it is meant that initial conditions in the plane $x0y$ are as following:

$$H_x(x,y,z=0)=0 \qquad H_y(x,y,z=0)=0 \qquad H_z(x,y,z=0)\neq 0 \tag{16}$$

It is follows from Eqs. (2) and (16), that

$$\left.\frac{\partial H_z(x,y,z)}{\partial z}\right|_{z=0}=0 \tag{17}$$

By the substituting Eq. (17) into Eqs. (10), (14) and (15), one can find that:

$$H_z(x,y,z) = \sum_{k=0}^{\infty}(-1)^k\frac{z^{2k}}{(2k)!}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^k H_z(x,y,z=0) \cong H_z(x,y,z=0) -\frac{z^2}{2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)H_z(x,y,z=0)+\cdots, \tag{18}$$

$$H_x(x,y,z) = \sum_{k=0}^{\infty}(-1)^k\frac{z^{2k+1}}{(2k+1)!}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^k \times\frac{\partial}{\partial x}H_z(x,y,z=0)\cong z\frac{\partial H_z(x,y,z=0)}{\partial x} -\frac{z^3}{6}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\frac{\partial H_z(x,y,z=0)}{\partial x}+\cdots \tag{19}$$

and

$$\begin{aligned} &H_y\left(x,y,z\right) \qquad (20) \\ &= \sum_{k=0}^{\infty} \left(-1\right)^k \frac{z^{2k+1}}{(2k+1)!} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^k \\ &\times \frac{\partial}{\partial y} H_z\left(x,y,z=0\right) \cong z \frac{\partial H_z\left(x,y,z=0\right)}{\partial y} \\ &- \frac{z^3}{6} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \frac{\partial H_z\left(x,y,z=0\right)}{\partial y} + \cdots \end{aligned}$$

As we can see, all the field components are expressed through the $H_z\left(x,y,z=0\right)$ only.

4 CONCLUSION

Results presented here may have extensive applications in accelerator theory. It give the way to calculate, for example, the focusing properties of wiggler or dynamical aperture of accelerator. The derived here analytical expressions can be used not only for wigglers and undulators, but for quadrupoles or sextupoles also. The application of this method for concrete devices calls for further investigation.

5 REFERENCES

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