

MAGNETIC FIELD EXPANSION IN A BENT-SOLENOID CHANNEL*

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Abstract

Bent-solenoid channels are being considered for generating dispersions desired for emittance exchange between the transverse and longitudinal motions of a muon beam in a longitudinal ionization cooling channel. Correct field expansion around the reference orbit is important for reliable particle-tracking simulation as well as beam dynamics analysis. This paper reviews the magnetic field expansion for a bent-solenoid channel. In particular, ready-to-use formulas for the magnetic field and its potentials are given.

1 INTRODUCTION

Bent-solenoid channels are being studied for longitudinal ionization cooling of a muon beam [1] in which the strong longitudinal magnetic field from the solenoid provides transverse focusing and the superimposed transverse dipole field provides dispersions desired for emittance exchange. In addition, a quadrupole field may be superimposed, for example, to achieve symmetric focusing. To study the beam dynamics in a bent-solenoid channel, magnetic field expansion around the reference orbit is needed. Although general formalism for such expansion is well-established, a reliable, ready-to-use result for bent solenoid seems not available. Furthermore, many errors, from “typos” to technical mistakes, exist in the literature on field expansion, as well as in computer code currently in use. In this paper, following the approach presented in ref. [2], we briefly review the derivation of field expansion and present *ready-to-use* expressions for the magnetic field and the corresponding scalar and vector potentials up to the octupole order.

2 GENERAL METHOD

The static magnetic field \mathbf{B} in a beam pipe must satisfy the source-free Maxwell equations in vacuum $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{B} = 0$. It is well known that both a scalar potential ϕ and a vector potential \mathbf{A} can be defined such that $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{B} = -\nabla\phi$. To study particle transport in a beam line, it is necessary to have a general field expression that depends on *free* parameters yet satisfies Maxwell equations in the neighborhood of a given design orbit.

2.1 Frenet-Serret Coordinate System

The Frenet-Serret coordinate system is commonly used to describe a particle’s motion around a design orbit. Its

base vectors are defined by the unit vectors $(\mathbf{e}_s, \mathbf{e}_x, \mathbf{e}_y)$ of the moving trihedron that can be determined from the Frenet formulas

$$\mathbf{e}'_s = -\kappa \mathbf{e}_x, \quad \mathbf{e}'_x = \kappa \mathbf{e}_s + \tau \mathbf{e}_y, \quad \mathbf{e}'_y = -\tau \mathbf{e}_x, \quad (1)$$

where the prime indicates derivative with respect to the path length s along the reference orbit, and $\kappa(s)$ and $\tau(s)$ are the local curvature and torsion of the reference orbit. (Note that, \mathbf{e}_x and \mathbf{e}_y are the opposites of the principal normal and binormal vectors commonly used in curve theory.) We consider the most common case that the reference orbit is a well-behaved plane (torsion-free) curve: $\tau(s) = 0$.

In the Frenet-Serret coordinate system, a particle’s position $\mathbf{r}(s)$ can be written as

$$\mathbf{r}(s) = \mathbf{r}_0(s) + x\mathbf{e}_x + y\mathbf{e}_y, \quad (2)$$

where $\mathbf{r}_0(s)$ is the reference orbit, and x and y are the transverse displacements from it. A vector field $\mathbf{A}(x, y, s)$ can be decomposed as

$$\mathbf{A} = A_x \mathbf{e}_x + A_y \mathbf{e}_y + A_s \mathbf{e}_s. \quad (3)$$

Note that, in the literature, A_s may represent the canonical component $h \mathbf{A} \cdot \mathbf{e}_s$ instead of $\mathbf{A} \cdot \mathbf{e}_s$.

Vector analysis using the Frenet-Serret coordinates can be carried out via

$$\nabla \phi = \frac{\partial \phi}{\partial x} \mathbf{e}_x + \frac{\partial \phi}{\partial y} \mathbf{e}_y + \frac{1}{h} \frac{\partial \phi}{\partial s} \mathbf{e}_s, \quad (4)$$

$$\nabla \cdot \mathbf{A} = \frac{1}{h} \frac{\partial (h A_x)}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{1}{h} \frac{\partial A_s}{\partial s}, \quad (5)$$

$$\begin{aligned} \nabla \times \mathbf{A} = & \left[\frac{\partial A_s}{\partial y} - \frac{1}{h} \frac{\partial A_y}{\partial s} \right] \mathbf{e}_x + \frac{1}{h} \left[\frac{\partial A_x}{\partial s} - \frac{\partial (h A_s)}{\partial x} \right] \mathbf{e}_y \\ & + \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right] \mathbf{e}_s, \end{aligned} \quad (6)$$

where the so-called scale factor, $h = 1 + \kappa(s)x$.

2.2 Scalar Potential and Recursion Formula

Expanding the magnetic field and its potentials as a series of x and y , a general magnetic field can be specified by the expansion coefficients as functions of s . Because the field must satisfy the Maxwell equations, those expansion coefficients are interrelated. An important exercise is to identify coefficients that can be chosen freely and express the field as a function of them.

It is convenient to start with the scalar potential $\phi(x, y, s)$. Using the Taylor series, ϕ can be expanded as

$$\phi(x, y, s) = - \sum_{n=0}^{\infty} \sum_{m=0}^n a_{n-m, m}(s) \frac{x^{n-m}}{(n-m)!} \frac{y^m}{m!}. \quad (7)$$

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The Maxwell equations are satisfied if and only if the scalar potential satisfies the Laplace equation $\nabla^2\phi = 0$. Working out $\nabla^2\phi$ using Eqs. (7, 4, 5) and expanding the result into a Taylor series of x and y , then requiring the coefficients to be zero, gives the recursive formula that the coefficient functions must satisfy

$$a_{n,m+2} = -a_{n+2,m} - \sum_{p=0}^n (-\kappa)^p \frac{n!}{(n-p)!} \left[\kappa a_{n-p+1,m} + (p+1)a''_{n-p,m} - \binom{p+2}{2} (n-p)\kappa' a'_{n-p-1,m} \right]. \quad (8)$$

Although rather obscure, this formula clearly shows that, corresponding to the even and the odd m 's, there are two independent sets of recursion relations for $a_{n,m}$'s. Because the second index m specifies the power of y in the potential expansion, the odd (even) set of coefficients defines the normal (skew) field component whose potential is anti-symmetric (symmetric) to the orbit plane. Furthermore, only $a_{n,0}(s)$ and $a_{n,1}(s)$ can be independently chosen. All higher-order (in m) coefficients can be derived from them. Using these independent coefficients and the recursion formula, it is straightforward, though tedious, to express the scalar potential and the magnetic field as functionals of free functions that characterize the field.

2.3 Independent Magnetic Field Components

The transverse components of the field on the reference plane and the longitudinal component on the reference orbit are determined by the independent coefficients $a_{n,0}$ and $a_{n,1}$ via

$$\text{skew : } B_x(x, 0, s) = \sum_{n=1}^{\infty} a_{n,0}(s) \frac{x^{n-1}}{(n-1)!}, \quad (9)$$

$$\text{normal : } B_y(x, 0, s) = \sum_{n=0}^{\infty} a_{n,1}(s) \frac{x^n}{n!}, \quad (10)$$

$$\text{longitudinal : } B_s(0, 0, s) = a'_{0,0}(s). \quad (11)$$

Conversely, these three field components completely determine the independent coefficients and the 3D magnetic field. Furthermore, these field components can be freely chosen.

In terms of standard (US convention) multipole coefficients a_n and b_n defined by [3]

$$B_y + iB_x = \sum_{n=0}^{\infty} (b_n + ia_n)(x + iy)^n, \quad (12)$$

the independent coefficients of the transverse field are

$$a_{n,1} = \left. \frac{\partial^n B_y}{\partial x^n} \right|_{x=y=0} = n! b_n, \quad (13)$$

$$a_{n,0} = \left. \frac{\partial^{n-1} B_x}{\partial x^{n-1}} \right|_{x=y=0} = (n-1)! a_{n-1}, \quad (14)$$

$$a'_{0,0} \equiv b_s \quad (15)$$

for the normal, skew, and longitudinal components, respectively. Note that we used the absolute multipole strengths instead of normalizing them by the dipole strength B_0 . Up to the octupole order, the independent coefficient functions are $b_s(s)$ for the solenoid component, $b_0(s)$ for the dipole, $b_1(s)$ and $a_1(s)$ for the normal and skew quadrupoles, $b_2(s)$ and $a_2(s)$ for the sextupoles, and $b_3(s)$ and $a_3(s)$ for the octupoles.

2.4 Vector Potential

The vector potential \mathbf{A} is important in Hamiltonian treatment of beam dynamics. To obtain a unique expression of the vector potential from the scalar potential ϕ , we choose the gauge

$$xA_x + yA_y = 0 \quad (16)$$

instead of the usual Coulomb gauge $\nabla \cdot \mathbf{A} = 0$. Using this gauge, we can introduce two new functions, $F(x, y, s)$ and $G(x, y, s)$, to express the three components of the vector potential as

$$A_x = -yF, \quad A_y = xF, \quad A_s = \frac{G}{h}. \quad (17)$$

Then the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ becomes

$$B_x = \frac{1}{h} \left(\frac{\partial G}{\partial y} - x \frac{\partial F}{\partial s} \right), \quad (18)$$

$$B_y = \frac{1}{h} \left(-\frac{\partial G}{\partial x} - y \frac{\partial F}{\partial s} \right), \quad (19)$$

$$B_s = x \frac{\partial F}{\partial x} + y \frac{\partial F}{\partial y} + 2F, \quad (20)$$

or

$$x \frac{\partial G}{\partial x} + y \frac{\partial G}{\partial y} = h (yB_x - xB_y), \quad (21)$$

$$x \frac{\partial F}{\partial x} + y \frac{\partial F}{\partial y} + 2F = B_s. \quad (22)$$

The equations for G and F are decoupled and can be solved separately via Euler's theorem for homogenous forms. It is easy to verify that their solutions can be written as

$$F = \sum_{n=0}^{\infty} \frac{1}{n+2} B_s^{(n)}, \quad (23)$$

$$G = \sum_{n=1}^{\infty} \left(\frac{1}{n} + \frac{\kappa x}{n+1} \right) (yB_x^{(n-1)} - xB_y^{(n-1)}). \quad (24)$$

Here the superscript (n) means the n th order component. Using the F and G expressions and Eq. (17), the vector potential can be straightforwardly worked out from the magnetic field expansion. Here we will not spell out the expressions as functions of $a_{n,m}$. (Note that the last expression for F in ref. [2] is incorrect.)

3 READY-TO-USE FIELD EXPANSION

In the previous section, we outlined the procedure to obtain the Taylor expansion of a magnetic field and its scalar and vector potentials. Although straightforward in principle, it is a rather tedious and fallible exercise to obtain the final expansions, especially with the dependent coefficients removed. With the help of Mathematica, we did this exercise for the bent-solenoid channels considered for the ionization cooling of a muon beam. We report the ready-to-use formulas that include all field components up to the octupole order, except for the horizontal dipole component $a_0(s)$, which makes the zeroth-order orbit a nonplanar curve. If the zeroth-order orbit is chosen as a reference, $\kappa(s) = qb_0(s)/p_s$, where q and p_s are the charge and longitudinal momentum. The bending radius $\rho(s) = |1/\kappa|$.

$$\begin{aligned} \phi(x, y, s) = & a_{0,0} + b_0 y \\ & + \frac{1}{2} a_1 x^2 + b_1 xy - \frac{1}{2} (a_1 + b'_s) y^2 \\ & + \frac{1}{3} a_2 x^3 + b_2 x^2 y - \frac{1}{2} [2a_2 + \kappa(a_1 - 2b'_s) - \kappa' b_s] xy^2 \\ & - \frac{1}{6} (2b_2 + b''_0 + \kappa b_1) y^3 + \frac{1}{4} a_3 x^4 + b_3 x^3 y \\ & - \frac{1}{4} [6a_3 + a''_1 + 2\kappa(a_2 + 3\kappa' b_s) - 2\kappa^2(a_1 - 3b'_s)] x^2 y^2 \\ & - \frac{1}{6} [6b_3 + b''_1 + 2\kappa(b_2 - b''_0) - \kappa^2 b_1 - \kappa' b'_0] xy^3 \\ & + \frac{1}{24} [6a_3 + 2a''_1 + b'''_s + \kappa(4a_2 + 5\kappa' b_s) - \kappa^2(a_1 - 4b'_s)] y^4 \end{aligned} \quad (25)$$

$$\begin{aligned} B_x(x, y, s) = & a_1 x + b_1 y + a_2 x^2 + 2b_2 xy \\ & - \frac{1}{2} [2a_2 + \kappa(a_1 - 2b'_s) - \kappa' b_s] y^2 + a_3 x^3 + 3b_3 x^2 y \\ & - \frac{1}{2} [6a_3 + a''_1 + 2\kappa(a_2 + 3\kappa' b_s) - 2\kappa^2(a_1 - 3b'_s)] xy^2 \\ & - \frac{1}{6} [6b_3 + b''_1 + 2\kappa(b_2 - b''_0) - \kappa^2 b_1 - \kappa' b'_0] y^3 \end{aligned} \quad (26)$$

$$\begin{aligned} B_y(x, y, s) = & b_0 + b_1 x - (a_1 + b'_s) y \\ & + b_2 x^2 - [2a_2 + \kappa(a_1 - 2b'_s) - \kappa' b_s] xy \\ & - \frac{1}{2} (2b_2 + b''_0 + \kappa b_1) y^2 + b_3 x^3 \\ & - \frac{1}{2} [6a_3 + a''_1 + 2\kappa(a_2 + 3\kappa' b_s) - 2\kappa^2(a_1 - 3b'_s)] x^2 y \\ & - \frac{1}{2} [6b_3 + b''_1 + 2\kappa(b_2 - b''_0) - \kappa^2 b_1 - \kappa' b'_0] xy^2 \\ & + \frac{1}{6} [6a_3 + 2a''_1 + b'''_s + \kappa(4a_2 + 5\kappa' b_s) - \kappa^2(a_1 - 4b'_s)] y^3 \end{aligned} \quad (27)$$

$$\begin{aligned} B_s(x, y, s) = & b_s - \kappa b_s x + b'_0 y \\ & + \frac{1}{2} (a'_1 + 2\kappa^2 b_s) x^2 + (b'_1 - \kappa b'_0) xy - \frac{1}{2} (a'_1 + b''_s) y^2 \\ & + \frac{1}{6} (2a'_2 - 3\kappa a'_1 - 6\kappa^3 b_s) x^3 + (b'_2 - \kappa b'_1 + \kappa^2 b'_0) x^2 y \end{aligned} \quad (28)$$

$$\begin{aligned} & - \frac{1}{2} [2a'_2 - 3\kappa b''_s + \kappa'(a_1 - 3b'_s) - \kappa'' b_s] xy^2 \\ & - \frac{1}{6} (2b'_2 + b'''_0 + \kappa b'_1 + \kappa' b_1) y^3 \end{aligned}$$

$$\begin{aligned} A_x(x, y, s) = & -\frac{1}{2} b_s y + \frac{1}{3} \kappa b_s xy - \frac{1}{3} b'_0 y^2 \\ & - \frac{1}{8} (2\kappa^2 b_s + a'_1) x^2 y - \frac{1}{4} (b'_1 - \kappa b'_0) xy^2 + \frac{1}{8} (a'_1 + b''_s) y^3 \\ & - \frac{1}{30} (2a'_2 - 3\kappa a'_1 - 6\kappa^3 b_s) x^3 y - \frac{1}{5} (b'_2 - \kappa b'_1 + \kappa^2 b'_0) x^2 y^2 \\ & + \frac{1}{10} [2a'_2 - 3\kappa b''_s + \kappa'(a_1 - 3b'_s) - \kappa'' b_s] xy^3 \\ & + \frac{1}{30} (2b'_2 + b'''_0 + \kappa b'_1 + \kappa' b_1) y^4 \end{aligned} \quad (29)$$

$$\begin{aligned} A_y(x, y, s) = & \frac{1}{2} b_s x - \frac{1}{3} \kappa b_s x^2 + \frac{1}{3} b'_0 xy \\ & + \frac{1}{8} (2\kappa^2 b_s + a'_1) x^3 + \frac{1}{4} (b'_1 - \kappa b'_0) x^2 y - \frac{1}{8} (a'_1 + b''_s) xy^2 \\ & + \frac{1}{30} (2a'_2 - 3\kappa a'_1 - 6\kappa^3 b_s) x^4 + \frac{1}{5} (b'_2 - \kappa b'_1 + \kappa^2 b'_0) x^3 y \\ & - \frac{1}{10} [2a'_2 - 3\kappa b''_s + \kappa'(a_1 - 3b'_s) - \kappa'' b_s] x^2 y^2 \\ & - \frac{1}{30} (2b'_2 + b'''_0 + \kappa b'_1 + \kappa' b_1) xy^3 \end{aligned} \quad (30)$$

$$\begin{aligned} A_s(x, y, s) = & -b_0 x \\ & - \frac{1}{2} (b_1 - \kappa b_0) x^2 + \frac{1}{2} (2a_1 + b'_s) xy + \frac{1}{2} b_1 y^2 \\ & - \frac{1}{6} (2b_2 - \kappa b_1 + 3\kappa^2 b_0) x^3 + \frac{1}{6} (6a_2 - 5\kappa b'_s - 2\kappa' b_s) x^2 y \\ & + \frac{1}{6} (6b_2 + b''_0) xy^2 - \frac{1}{6} [2a_2 + \kappa(a_1 - 2b'_s) - \kappa' b_s] y^3 \\ & - \frac{1}{12} (3b_3 - \kappa b_2 + 2\kappa^2 b_1 - 6\kappa^3 b_0) x^4 \\ & + \frac{1}{24} (24a_3 + 3a''_1 + 26\kappa^2 b'_s + 20\kappa\kappa' b_s) x^3 y \\ & + \frac{1}{24} (36b_3 + 3b''_1 - 7\kappa b''_0 - 3\kappa' b'_0) x^2 y^2 \\ & - \frac{1}{24} [24a_3 + 5a''_1 + b'''_s + 8\kappa(a_2 + 3\kappa' b_s) \\ & \quad - 8\kappa^2(a_1 - 3b'_s)] xy^3 \\ & - \frac{1}{24} [6b_3 + b''_1 + 2\kappa(b_2 - b''_0) - \kappa^2 b_1 - \kappa' b'_0] y^4 \end{aligned} \quad (31)$$

4 REFERENCES

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