INITIAL AND SUBSEQUENT GROWTH OF THE FAST ION ("ION HOSE") INSTABILITY*

R. A. Bosch,# Synchrotron Radiation Center, University of Wisconsin-Madison, 3731 Schneider Dr., Stoughton, WI 53589, USA

Abstract

The fast ion instability is considered for a distribution of ion bounce frequencies. Because of the spread in bounce frequencies, the instability initially grows exponentially with propagation distance. When the initial growth saturates, the instability grows exponentially with the square root of the propagation distance; the saturated growth equals that calculated when the spread in ion bounce frequencies is neglected. For a broad distribution of ion bounce frequencies, instability may be prevented by a betatron damping rate that exceeds the incoherent betatron frequency shift induced by ions at the tail of the bunch train.

1 INITIAL GROWTH

To model the initial growth of the fast ion instability [1–11] and thereby determine the betatron damping rate or feedback necessary to prevent it, a distribution of ion "bounce" frequencies is considered. The bounce frequency is the natural frequency of transverse ion oscillations about the electron orbit [12], given for small vertical oscillations ($<<\sigma_y$) of singly charged ions by

$$\omega_i = \left(\frac{n_e e^2}{\varepsilon_o m_i} \frac{\sigma_x}{\sigma_x + \sigma_y} \right)^{1/2} , \tag{1}$$

in which n_e is the time-averaged electron density on axis during the passage of a bunch train or electron beam, m_i is the ion mass, σ_x and σ_y are the horizontal and vertical beam dimensions, e is the electron charge and ε_o is the permittivity of free space. Because of the dependence upon the ion mass and the position-dependent quantities n_e, σ_x and σ_y, a large range of ion bounce frequencies may be expected in a typical electron storage ring.

We consider a magnetically focused electron beam or bunch train, using a smooth approximation for the betatron focusing. In the case of a bunch train, we model a bunch train of duration τ_b as an electron beam of duration τ_b. The propagation time $Z \equiv z/v$ describes the propagation distance z divided by beam velocity v, while coordinate $T \equiv t - z/v$ denotes the time after passage of the beam head. We assume that the ion density grows linearly with the passage of the beam. For small vertical displacements ($<<\sigma_y$) of the electrons, the electron "bounce frequency" in the ion channel, denoted $\omega_e(T)$, is

$$\omega_e(T) = \left(\frac{n_i(T) e^2}{\varepsilon_o \gamma m_e} \frac{\sigma_x}{\sigma_x + \sigma_y} \right)^{1/2} , \tag{2}$$

where $n_i(T)$ is the ion density at a time T after the passage of the beam head, m_e is the electron mass, and γ is the relativistic factor. For ions created by collision of the electrons with neutral molecules and lost on a time scale large compared to the beam duration τ_b, $n_i(T)$ is proportional to T, so we have

$$\omega_e^2(T) = KT , \tag{3}$$

where $K \equiv \omega_e^2(\tau_b)/\tau_b$.

The approximate equations of motion for the electron beam vertical position $b(Z,T)$, the average vertical position $c(Z,T)$ of all ions, and the average vertical position $c_i(Z,T)$ of those ions with bounce frequency ω_i are [1]

$$[\partial_Z^2 + \omega_\beta^2 + \omega_e^2(T)] b(Z,T) = \omega_e^2(T) c(Z,T) , \tag{4}$$

$$\begin{aligned} &\left[\frac{1}{T} \partial_T T \partial_T + \omega_i^2 \right] c_i(Z,T) \\ &= \left[\partial_T^2 + \frac{1}{T} \partial_T + \omega_i^2 \right] c_i(Z,T) = \omega_i^2 b(Z,T) , \end{aligned} \tag{5}$$

where ω_β is the electron betatron frequency in the absence of ions.

When $\omega_\beta = 0$, eqs. (1)–(5) describe the "ion hose" instability of an electron beam focused by a beam-induced ion channel [1]. In an electron storage ring, where ion effects are a perturbation to the betatron motion in applied magnetic fields, we instead have $\omega_\beta >> \omega_e(T)$.

In eq. (5), the term $(1/T)\partial_T c_i(Z,T)$ describes the damping of collective ion oscillations that results from the constant creation rate of stationary ions. This damping rate of ~$1/T$ is small compared to the phase-mix damping rate ~$\delta\omega_i$ from an ion frequency spread of $\delta\omega_i$ provided that $\delta\omega_i >> 1/T$. Consequently, its neglect is justified for $\delta\omega_i T >> 1$, i.e. several ion oscillation periods behind the head of the beam when there is a large ion frequency spread. For $\delta\omega_i T >> 1$, we therefore approximate eq. (5) as

$$[\partial_T^2 + \omega_i^2] c_i(Z,T) = \omega_i^2 b(Z,T) . \tag{6}$$

* Work supported by NSF grant DMR-0084402

Email: bosch@src.wisc.edu

For a disturbance originating at $(Z,T) = (0,0)$, we look for a solution where $b(Z,T)$ and $c(Z,T)$ are of the form [10]

$$b(Z,T) = b_o \exp\bigl(g(Z,T)\bigr)\exp\left[i\left(\omega_\beta + \frac{\omega_e^{\,2}(T)}{2\omega_\beta}\right)Z - i\omega T\right], \quad (7)$$

with $\omega > 0$. In eq. (7), the incoherent betatron frequency shift resulting from the ions, $\omega_e^2(T)/2\omega_\beta$, is included in the oscillation frequency. The slowly-varying function $g(Z,T)$ describes the oscillation growth. To ensure that ω is the initial oscillation frequency in the laboratory, we consider solutions where $g(Z,T)$ is real for small Z.

Substituting eq. (7) into eqs. (4) and (6) yields

$$\left(2i\omega_\beta \partial_Z g\right) b = KTc\,, \quad (8)$$

$$\left[-2i\tilde{\omega}(Z)\partial_T g - \tilde{\omega}^2(Z) + \omega_i^{\,2}\right] c_i = \omega_i^{\,2} b\,, \quad (9)$$

where $\tilde{\omega}(Z)$ is given by

$$\tilde{\omega}(Z) \equiv \omega - \frac{KZ}{2\omega_\beta}\,. \quad (10)$$

The quantity $\tilde{\omega}(Z)$ is the oscillation frequency in the laboratory given by eq. (7) when $g(Z,T)$ is real; $\tilde{\omega}(Z)$ equals ω for $Z = 0$, and equals zero for $Z = Z_o$, where

$$Z_{\mathrm{o}} \equiv \frac{2\omega\,\omega_\beta}{K}\ . \quad (11)$$

For a normalised distribution over positive ion bounce angular frequencies $f(\omega_i)$, the average ion vertical position, $c = \int c_i\, f(\omega_i)\, d\omega_i$ may be obtained from eq. (9), giving

$$c = b\int_0^\infty \frac{\omega_i^{\,2} f(\omega_i)\, d\omega_i}{-2i\tilde{\omega}(Z)\partial_T g - \tilde{\omega}^2(Z) + \omega_i^{\,2}}\,. \quad (12)$$

Using eq. (12) to eliminate $c(Z,T)$ from eq. (8) yields

$$\partial_Z g = \frac{KT}{2i\omega_\beta}\int_0^\infty \frac{\omega_i^{\,2} f(\omega_i)\, d\omega_i}{-2i\tilde{\omega}(Z)\partial_T g - \tilde{\omega}^2(Z) + \omega_i^{\,2}}\ . \quad (13)$$

To obtain the initial growth rate when $g(Z,T)$ is real, consider the weak growth limit ($\partial_T g \to 0$) given by the Plemelj formula [13]

$$\partial_Z g = \frac{KT}{2i\omega_\beta}\left[\begin{array}{l} \mathrm{PV}\displaystyle\int_0^\infty d\omega_i\, \frac{f(\omega_i)\omega_i^{\,2}}{\omega_i^{\,2} - \tilde{\omega}^2(Z)} \\ + i\dfrac{\pi}{2}\mathrm{sgn}\left[\mathrm{Re}(\partial_T g)\right]\tilde{\omega}(Z) f\left(\left|\tilde{\omega}(Z)\right|\right) \end{array}\right]. \quad (14)$$

For an ion distribution with width $\delta\omega_i$, $\tilde{\omega}(Z) \approx \omega$ and $f(|\tilde{\omega}(Z)|) \approx f(\omega)$ when $Z << (\delta\omega_i/\omega)Z_o$. Equation (14) then becomes

$$\partial_Z g = \frac{KT}{2i\omega_\beta}\left[\mathrm{PV}\int_0^\infty d\omega_i\,\frac{f(\omega_i)\omega_i^2}{\omega_i^{\,2} - \omega^2} + i\frac{\pi}{2}\mathrm{sgn}\left[\mathrm{Re}(\partial_T g)\right]\omega f(\omega)\right] \quad (15)$$

which integrates to give solutions with $g(0,T) = g(Z,0) = 0$

$$g(Z,T) = \frac{KTZ}{2i\omega_\beta}\left[\mathrm{PV}\int_0^\infty d\omega_i\, \frac{f(\omega_i)\omega_i^{\,2}}{\omega_i^{\,2} - \omega^2} \pm i\frac{\pi}{2}\omega\, f(\omega)\right]. \quad (16)$$

The requirement that $g(Z,T)$ be a real function for small Z determines ω, which obeys

$$\mathrm{PV}\int_0^\infty d\omega_i\, \frac{f(\omega_i)\omega_i^{\,2}}{\omega_i^{\,2} - \omega^2} = 0\ . \quad (17)$$

For an ion distribution peaked at ω_{io}, eq. (17) indicates that $\omega \approx \omega_{io}$. Thus, we have

$$g(Z,T) = \frac{KTZ}{2\omega_\beta}\left[\pm\frac{\pi}{2}\omega f(\omega)\right] \approx \frac{KTZ}{2\omega_\beta}\left[\pm\frac{\pi}{2}\omega_{io}\, f(\omega_{io})\right] \quad (18)$$

The solution with the "+" sign undergoes exponential growth in T and Z. Substituting eq. (18) into eq. (7) yields solutions valid for $Z << (\delta\omega_i/\omega_{io})Z_o$, $\delta\omega_i T >> 1$, and sufficiently small Z that the Plemelj formula applies

$$\begin{aligned} b(Z,T) = b_o \exp&\left(\frac{KTZ}{2\omega_\beta}\left[\pm\frac{\pi}{2}\omega_{io}\, f(\omega_{io})\right]\right) \\ &\times\exp\left[i\left(\omega_\beta + \frac{KT}{2\omega_\beta}\right)Z - i\omega_{io}T\right]. \end{aligned} \quad (19)$$

For a broad distribution of ion bounce frequencies where $\delta\omega_i/\omega_{io} \sim 1/2$, the initial growth rate in Z is comparable to the incoherent betatron frequency shift of $\omega_e^2(T)/2\omega_\beta$ that results from the ions. Consequently, instability may be prevented by a betatron damping rate that exceeds the incoherent betatron frequency shift induced by ions at the tail of the bunch train.

2 SUBSEQUENT GROWTH

Consider the instability growth when $(\delta\omega_i/\omega_{io})Z_o << Z << 2Z_o$ and $\delta\omega_i T >> 1$. For a disturbance with $\omega \approx \omega_{io}$, we have $\left\|\tilde{\omega}(Z)\right| - \omega_{io}\right| >> \delta\omega_i$, so that eq. (13) becomes

$$\partial_Z g = \frac{KT}{2i\omega_\beta}\left(\frac{\omega_{io}^{\,2}}{-2i\tilde{\omega}(Z)\partial_T g - \tilde{\omega}^2(Z) + \omega_{io}^{\,2}}\right). \quad (20)$$

Equation (20) is identical to that describing the fast ion instability when all ions have the same bounce frequency (see Ref. [10], eq. 18). When $Z << Z_o$, eq. (20) becomes

$$\left(\partial_Z g\right)\left(\partial_T g\right) = \frac{KT\omega_{io}}{4\omega_\beta}\ , \quad (21)$$

with solutions [2]

$$g(Z,T) = \pm\left(\frac{K\omega_{io}Z}{2\omega_\beta}\right)^{1/2} T + constant\,. \quad (22)$$

For $(\delta\omega_i/\omega_{io})Z_o << Z << Z_o$, the growing solution given by eq. (22) has lower values of $\partial_Z g$ and $\partial_T g$ than is given by eq. (18), indicating that the initial instability growth has saturated. The saturation occurs when the oscillation frequency in the laboratory, $\tilde{\omega}(Z)$, no longer coincides with a typical ion bounce frequency. The saturated growth equals that calculated when the spread in ion bounce frequencies is neglected.

3 EXAMPLE

Consider a Cauchy distribution of ion bounce frequencies with peak at ω_{io} and half-width $\delta\omega_i << \omega_{io}$

$$f(\omega_i) = \frac{\delta\omega_i / \pi}{(\omega_i - \omega_{io})^2 + (\delta\omega_i)^2} \quad . \tag{23}$$

For a Cauchy distribution, the decoherence function [3] defined by $\hat{D}(T) \equiv \int d\omega_i\, f(\omega_i) \exp[i(\omega_i - \omega_{io})T]$ equals $\exp(-\delta\omega_i T)$. The exponential ion decoherence (also called "phase-mix damping") for a Cauchy distribution of ion bounce frequencies behaves like frictional damping ("phenomenological" or "Q-damping") of ions with a single bounce frequency [14]. For an exponential ion decoherence, eq. (30) of Ref. [3] gives a solution for $g(Z,T)$ with $g(Z,0) = g(0,T) = 0$, valid for $Z << Z_o$. In our notation, this solution is

$$\begin{aligned}\exp[g(Z,T)] = A(Z,T) &= \exp(-\delta\omega_i T) I_o\left(\left[\frac{T^2 Z K \omega_{io}}{2\omega_\beta}\right]^{1/2}\right) \\ &+ \delta\omega_i \int_0^T dT' I_o\left(\left[\frac{(T^2 - T'^2) Z K \omega_{io}}{2\omega_\beta}\right]^{1/2}\right) \exp(-\delta\omega_i [T - T']),\end{aligned} \tag{24}$$

where I_o is the zeroth-order Bessel function of imaginary argument.

For small Z, the Taylor expansion $A(Z,T) = A(0,T) + [\partial A/\partial Z(0,T)]Z$ gives

$$A(Z,T) = 1 + \left[\frac{K\omega_{io} Z}{4\omega_\beta (\delta\omega_i)^2}\right][\exp(-\delta\omega_i T) - 1 + \delta\omega_i T], \tag{25}$$

which, for $\delta\omega_i T >> 1$, reduces to

$$\begin{aligned}\exp[g(Z,T)] = A(Z,T) &= 1 + \left[\frac{KTZ\omega_{io}}{4\omega_\beta (\delta\omega_i)}\right] \\ &\approx \exp\left[\frac{KTZ\omega_{io}}{4\omega_\beta (\delta\omega_i)}\right].\end{aligned} \tag{26}$$

This is identical to the growing solution given by eq. (18), confirming that eq. (18) describes the initial instability growth. For a Cauchy distribution of ion bounce frequencies, we expect that the Plemelj formula may be applied to eq. (13) when the integrand's pole at $\omega_{io} + i\partial_T g$ is much closer to the real axis than the poles of $f(\omega_i)$ located at $\omega_{io} \pm i\delta\omega_i$. Thus, eq. (26) is expected to apply for $Z << (\delta\omega_i/\omega_{io})^2 Z_o$ and $\delta\omega_i T >> 1$.

In contrast, for $Z >> Z_o/(\omega_{io}T)^2$, the Bessel functions in eq. (24) may be approximated by their large-argument asymptotic expansion, yielding eq. (31) of Ref. [3]

$$\begin{aligned}A(Z,T) = &\left[2\pi\left(\frac{T^2 Z K\omega_{io}}{2\omega_\beta}\right)^{1/2}\right]^{-1/2} \exp\left[\left(\frac{T^2 Z K\omega_{io}}{2\omega_\beta}\right)^{1/2} - \delta\omega_i T\right] \\ &\times\left\{1 + \left[\frac{\pi(\delta\omega_i T)^2}{2}\left(\frac{T^2 Z K\omega_{io}}{2\omega_\beta}\right)^{-1/2}\right]^{1/2} \exp\left[\frac{(\delta\omega_i T)^2}{2}\left(\frac{T^2 Z K\omega_{io}}{2\omega_\beta}\right)^{-1/2}\right]\right\}\end{aligned} \tag{27}$$

For $Z >> (\delta\omega_i/\omega_{io})^2 Z_o$, eq. (27) reproduces the growing solution given by eq. (22).

Consequently, for $(\delta\omega_i/\omega_{io})^2 Z_o << Z << Z_o$ and $\delta\omega_i T >> 1$, eq. (22) describes instability growth for a Cauchy distribution. As expected, eq. (22) describes the saturated growth subsequent to (i.e., downstream of) that described by eq. (18).

4 SUMMARY

We have considered the initial and subsequent growth of the fast ion instability for a distribution of ion bounce frequencies. The initial growth is exponential in Z and T, where Z is the propagation distance divided by beam velocity, and T the time elapsed since the head of the beam has passed. For larger Z, the growth is exponential in $Z^{1/2}$ and T; this saturated growth equals that calculated when the spread in ion bounce frequencies is neglected. For a broad distribution of ion bounce frequencies, instability growth may be prevented by a betatron damping rate that exceeds the incoherent betatron frequency shift induced by ions at the tail of the bunch train.

REFERENCES

[1] K. T. Nguyen, R. F. Schneider, J. R. Smith and H. S. Uhm, Appl. Phys. Lett. **50**, 239 (1987).

[2] T. O. Raubenheimer and F. Zimmerman, Phys. Rev. E **52**, 5487 (1995).

[3] G. V. Stupakov, T. O. Raubenheimer and F. Zimmermann, Phys. Rev. E **52**, 5499 (1995).

[4] K. Ohmi, Phys. Rev. E **55**, 7550 (1997).

[5] J. Byrd, A. Chao, S. Heifets, M. Minty, T. O. Raubenheimer, J. Seeman, G. Stupakov, J. Thomson and F. Zimmermann, Phys. Rev. Lett. **79,** 79 (1997).

[6] S. Heifets in *Proceedings of the 1997 Particle Accelerator Conference, Vancouver, Canada* (IEEE, Piscataway, NJ, 1998), p. 1620.

[7] J. Y. Huang, M. Kwon, T.-Y. Lee, I. S. Ko, Y. H. Chin and H. Fukuma, Phys. Rev. Lett. **81**, 4388 (1998).

[8] D. V. Pestrikov, Phys. Rev. ST Accel. Beams **2**, 044403 (1999).

[9] G. V. Stupakov, Phys. Rev. ST Accel. Beams **3**, 019401 (2000).

[10] R. A. Bosch, Phys. Rev. ST Accel. Beams **3**, 034402 (2000).

[11] G. J. Caporaso and J. F. McCarrick in *Proceedings of the 20th International Linac Conference, Monterey, 2000,* edited by A. W. Chao, eConf C000821, p. 500 (2000).

[12] E. Keil and B. Zotter, CERN-ISR-TH/71-58 (1971).

[13] D. R. Nicholson, *Introduction to Plasma Theory* (Wiley, New York, 1983).

[14] R. A. Bosch and R. M. Gilgenbach, Phys. Fluids **31**, 2006 (1988).