# High-Frequency Impedance of Small-Angle Collimators

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*Abstract*

We give a review of possible parameter regimes for impedance of both round and rectangular tapered collimators based on the physical mechanism involved in formation of the beam wake field. A simple analytical method is presented for impedance calculation in the diffraction regime. Theoretical results are compared with recent measurements of collimator wake at SLAC.

## 1 INTRODUCTION

Collimators are often used in storage rings and accelerators to remove high-amplitude particles from the transverse profile of the beam. Being close to the beam orbit they may introduce large impedance that perturbs the beam motion downstream of the collimator and results in additional emittance growth and jitter amplification of the collimated beam. The wake effect of collimators is of concern for future colliders, such as the Next Linear Collider [1], with extremely small transverse emittance of the beam.

To lower the collimator impedance one can try to taper the collimator jaws to get a gradual transition from a large to a small aperture and back. Two examples of such collimators – a round and a rectangular one – are shown in Fig. 1. We assume that the characteristic angle of the transition $\alpha$ is small, $\alpha \ll 1$.

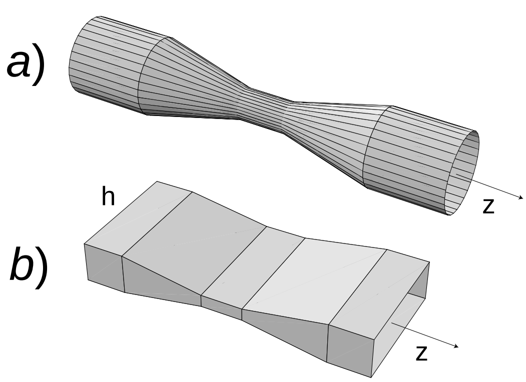


Figure 1: Geometry of a round (*a*) and a rectangular (*b*) tapered collimators. For the rectangular collimator, $h$ denotes its width in the horizontal plane.

The impedance of a smooth round tapered transition was calculated by K. Yokoya in the limit of low frequencies [2]. For the transverse impedance, Yokoya's formula gives

$$Z_t = -\frac{iZ_0}{2\pi}\int dz \left(\frac{b'}{b}\right)^2, \tag{1}$$

where $b(z)$ is the radius of the collimator as a function of the longitudinal coordinate $z$, $Z_0 = 377$ Ohm, and the prime denotes the derivative with respect to $z$. Note that $\alpha \approx |b'|$. It was shown later [3] that the condition of applicability of Yokoya's formula is

$$\alpha k b \ll A, \tag{2}$$

where $\alpha$ is the angle of the collimator, $b$ is the minimal radius, $k = \omega/c$, and $A$ is a numerical factor of order of unity. For a bunch of length $\sigma_z$, the characteristic value of $k$ in the beam spectrum is equal to $\sigma_z^{-1}$.

A generalization of Yokoya's result for a rectangular collimator of large aspect ratio, $h \gg b$, was given in Ref. [4]. The vertical transverse impedance in this case is

$$Z_t = -\frac{iZ_0 h}{2}\int dz \frac{(b')^2}{b^3}, \tag{3}$$

where $b(z)$ is the half-gap of the collimator. Eq. (3) shows that the impedance of a wide flat collimator may be much larger than the impedance of the round collimator with the same gap $b(z)$ – a result that seems surprising if one considers the limit $h \to \infty$. It turns out however, that the applicability condition of Eq. (3) is

$$\alpha k h^2/b \ll 1. \tag{4}$$

It is much tighter than Eq. (2) and means that Eq. (3) is not applicable for very large values of $h$. Unfortunately, the condition (4) was not formulated in the original publication [4], and in some cases resulted in the use of Eq. (3) beyond its applicability limits.

## 2 CLASSIFICATION OF POSSIBLE REGIMES

Consider a beam propagating in a straight round perfectly conducting pipe of a constant radius. For impedance calculations, it is convenient to assume a Fourier transform of the beam current $I_\omega = I_0 e^{ikz-i\omega t}$, where $I_0$ is the amplitude of the current. For an ultrarelativistic beam considered in this paper $k = \omega/c$. It is the image current in the wall that can produce electromagnetic radiation in the pipe, because the beam itself moves along a straight line with a constant velocity. The current density $j_{\rm im}$ of the image current will have the same dependance $j_{\rm im} \sim e^{ikz-i\omega t}$, which means that the image current propagates with the speed of light. Since electromagnetic waves in a straight pipe have a phase velocity greater than the speed of light, the wall currents do not excite these modes, hence, the beam does not radiate. This explains why impedance vanishes in a straight perfectly conducting pipe.

Now, assume a round tapered collimator of length $l \sim b/\alpha$ with a perfectly conducting wall. Because of the variation of the pipe radius, the image current acquires an additional factor $f(z)$, $j_{\rm im} \sim f(z)e^{ikz-i\omega t}$, where $f(z)$ is a smooth function varying on the scale of variation $l$ of the taper radius. Making Fourier decomposition of $j_{\rm im}$ with respect to $z$, we find the spectrum of width $\Delta k \sim l^{-1}$. Due to this spread in $k$ in the spectrum of $j_{\rm im}$ for the same frequency $\omega$, there will be harmonics with the phase velocity $v_{\rm ph}$ exceeding the speed of light. It is easy to see that by order of magnitude $|v_{\rm ph} - c| \sim c^2\Delta k/\omega \sim c/lk$. If the phase velocity of the image current becomes equal to the phase velocity of an eigenmode, the mode can be excited. Comparing $v_{\rm ph}$ with the lowest phase velocity $c(1 + j_{01}^2/k^2b^2)^{1/2}$ of TM modes in a round pipe of radius $b$ ($j_{01}$ is the first root of the Bessel function $J_0$), we find that the radiation begins if

$$kb^2/l = kb\alpha \gtrsim j_{01}^2. \qquad (5)$$

This condition, by order of magnitude, is opposite to Eq. (2) which explains why the impedance in Eq. (1) is purely inductive — in the regime given by Eq. (2) the beam does not radiate, and hence, does not lose energy.

We will call the regime of parameters where Eq. (5) is satisfied the *diffraction* regime.

The same consideration applies to the rectangular collimator with a large aspect ratio. An important difference, however, is that the lowest phase velocity of $\mathrm{TE}_{0n}$ eigenmodes in a rectangular waveguide is equal to $c(1 + \pi^2/k^2h^2)^{1/2}$. Hence, the radiation occurs when

$$kh^2/l = kh^2\alpha/b \gtrsim \pi^2. \qquad (6)$$

When this equation is satisfied the regime of purely inductive impedance Eq. (3) breaks down. This is in agreement with the fact that the applicability condition Eq. (4) is opposite to Eq. (6). If the angle of the collimator is such that Eq. (6) is satisfied, but $kb\alpha \ll 1$, only a limited number of $\mathrm{TE}_{0n}$ modes will be excited. We will call this regime *intermediate* to indicate that it is located in the parameter space between the inductive regime and the diffraction one. When $kb\alpha$ is greater than unity, the rectangular collimator is in the diffraction regime, and one can expect the wake of the rectangular collimator similar to the wake of the round collimator in the diffraction regime.

Below we will give a quick recipe for calculation of the diffraction impedance for both round and flat collimators. A detailed derivation based on solution of Maxwell's equation can be found in Ref. [5].

## 3 DIFFRACTION REGIME

### 3.1 Round collimator

Consider first a round collimator with the minimal collimator radius $b_1$. The pipe radius outside of the collimator is equal to $b_2$, $b_2 > b_1$.

The transverse wake is due to the excitation of dipole modes in the waveguide. These modes are excited by the dipole momentum of the beam, and instead of considering a current with an offset $\Delta$, we can consider two currents of opposite sign, $I_0$ and $-I_0$, with the offsets $\Delta/2$ and $-\Delta/2$, respectively (we omit the factor $e^{-i\omega t+ikz}$ in what follows). The electric field of such a dipole current in the pipe of radius $b_2$ is

$$\boldsymbol{E}^0 = \frac{2I_0\Delta}{c}\left[\frac{2\boldsymbol{\rho}(\hat{\boldsymbol{y}}\boldsymbol{\rho}) - \rho^2\hat{\boldsymbol{y}}}{\rho^4} + \frac{\hat{\boldsymbol{y}}}{b_2^2}\right], \qquad (7)$$

where $\hat{\boldsymbol{y}}$ is the unit vector in the direction of the offset and $\rho$ is the radius in the cylindrical coordinate system. The first term in the square brackets is the vacuum dipole field and the second term is the field generated by the image charges in the pipe.

To calculate the transverse impedance, we first find the energy lost by the beam after passage of the first taper of the collimator. In diffraction regime, the beam field in the annulus $b_1 < \rho < b_2$ will be cut off and reflected by the lateral walls of the taper. The energy flow $P_\omega$ in this region will be transferred into the propagating eigenmodes of the waveguide, and lost for the beam. To find $P_\omega$ we first calculate the average Poynting vector,

$$S = \frac{c}{8\pi}(\boldsymbol{E}^0)^2 = \frac{I_0^2\Delta^2}{2\pi c}\left(\frac{1}{b^4} + \frac{1}{b_2^4}\right), \qquad (8)$$

where $\boldsymbol{E}^0$ is given by Eq. (7), and then integrate it over the pipe cross section between $b_1$ and $b_2$:

$$P_\omega = \int_{b_1}^{b_2} 2\pi\rho d\rho S = \frac{I_0^2\Delta^2}{2b_1^2 c}\left(1 - \frac{b_1^4}{b_2^4}\right). \qquad (9)$$

The ratio $2P_\omega/I_0^2$ gives the real part of the longitudinal dipole impedance $\mathrm{Re}\,Z_l$, which, using the Panofsky-Wentzel theorem, can be related to the real part of the transverse impedance:

$$\mathrm{Re}\,Z_t = 2\frac{1}{k\Delta}\frac{\partial \mathrm{Re}\,Z_l}{\partial\Delta} = \frac{4(1 - b_1^4/b_2^4)}{\omega b_1^2}, \qquad (10)$$

where an additional factor of 2 now takes into account the contribution of both tapers of the collimator.

Knowledge of the real part of the transverse impedance allows us to calculate the kick factor $\kappa$ for a Gaussian beam. The kick factor is defined so that $Nr_e\kappa y_0/\gamma$ gives the deflection angle for the bunch, with $N$ being the number of particles in the bunch, $r_e$ – the classical electron radius, and $y_0$ – the beam offset. Using a result of Ref. [3] we can express $\kappa$ in terms of $\mathrm{Re}\,Z_t$,

$$\kappa = \int_0^\infty d\omega F\left(\frac{\omega\sigma_z}{c}\right)\mathrm{Re}\,Z_t(\omega), \qquad (11)$$

where $F(x) = -(i/\pi)e^{-x^2}\mathrm{erf}(ix)$. Putting Eq. (10) in Eq. (11) and performing integration, we find for the round collimator

$$\kappa = \frac{2(1 - b_1^4/b_2^4)}{b_1^2}. \qquad (12)$$

### 3.2 Rectangular collimator

For a rectangular collimator, an analog of Eq. (7) would be the field of a dipole current in a rectangular pipe of cross-section $2b_2 \times 2h$. We will simplify the problem, however, assuming that $h, b_2 \gg b_1$ and taking for $\boldsymbol{E}_0$ the vacuum dipole field of an offset current

$$E_x^0 = \frac{4I_0\Delta}{c}\frac{xy}{(x^2+y^2)^2}, \; E_y^0 = \frac{4I_0\Delta}{c}\frac{y^2-x^2}{(x^2+y^2)^2}, \quad (13)$$

where $y$ is the vertical, and $x$ is the horizontal coordinates. During the passage through the collimator, the beam field is now "scraped off" in the area $b_1 < |y| < b_2$ and since $b_2$ is assumed large, we can take the limit $b_2 \to \infty$. The power lost by the beam for a single taper is

$$P_\omega = 2\int_{b_1}^{\infty} dy \int_{-\infty}^{\infty} \frac{c}{8\pi}[(E_x^0)^2 + (E_y^0)^2]dx. \quad (14)$$

The integration yields

$$P_\omega = \Delta^2 I_0^2/4cb_1^2, \quad (15)$$

which is exactly half of the result Eq. (9) for the round collimator in the limit $b_2 \gg b_1$.

We conclude, that the impedance (kick factor) of the flat collimator in the diffraction regime is equal to half of the impedance (kick factor) for the round collimator with the same minimal gap $b_1$.

## 4 FLAT COLLIMATOR, INTERMEDIATE REGIME

The real part of the transverse impedance of a rectangular collimator in the intermediate regime is $\mathrm{Re}\, Z_t = 8\sqrt{\pi}\alpha^{1/2}/3k^{1/2}b_1^{3/2}c$ (see Ref. [5]). The kick factor calculated with the use of Eq. (11) gives

$$\kappa = 2.7\frac{\alpha^{1/2}}{\sigma_z^{1/2}b_1^{3/2}}. \quad (16)$$

## 5 REVIEW OF DIFFERENT REGIMES

The above theoretical results were obtained for limiting cases, when the appropriate parameters are much smaller or much larger than unity, see Eqs. (2), (4), (5) and (6). In an attempt to define more precisely the boundary between different regimes, we used the following approach. We found the condition at which the kick factors given by two adjacent regimes become equal and assumed that the value of the parameter that exactly satisfies this condition defines the boundary between the regimes. The equations for kick factors $\kappa$ with the so defined applicability conditions are given below for the case $b_2 \gg b_1$.

For round collimator,

$$\kappa = \begin{cases} 2/b_1^2 & \text{for} \quad \alpha b_1/\sigma_z > 2\sqrt{\pi}, \\ \alpha/\sqrt{\pi}\sigma_z b_1 & \text{for} \quad \alpha b_1/\sigma_z < 2\sqrt{\pi}. \end{cases} \quad (17)$$

For rectangular collimators with large aspect ratio, $h \gg b_1$,

$$\kappa = \begin{cases} 1/b_1^2 & \text{for} \quad \sqrt{\alpha b_1/\sigma_z} > 0.37, \\ 2.7\sqrt{\alpha/\sigma_z b_1^3} & \text{for} \quad 0.37 > \sqrt{\frac{\alpha b_1}{\sigma_z}} > 3.1\frac{b_1}{h}, \\ \sqrt{\pi}\alpha h/2\sigma_z b_1^2 & \text{for} \quad \sqrt{\alpha b_1/\sigma_z} < 3.1 b_1/h. \end{cases} \quad (18)$$

One should not expect a good accuracy from the above formulae when applied in the transition regions.

## 6 COMPARISON WITH EXPERIMENT

In the experiment at SLAC linac, four different apertures were used with varying angles and gaps (description of the apparatus and details of the measurement can be found in Ref. [6]). Three apertures satisfied the condition $h \gg b_1$, and one of the collimators had a square cross section. The comparison between theoretically computed and experimentally measured kick factors $\kappa$ is given in Table 1. For calculation of the wake for the square collimator, we used Eqs. (17) derived for the round geometry, and for rectangular collimators we used Eqs. (18). Generally, the agreement is within a factor of two. One has to keep in mind, though, that the parameters of the experiment were close to the boundary between the diffraction and inductive or intermediate regimes, where the theoretical formulas are not as accurate as in limiting cases.

Table 1: Comparison of the measured and calculated kick factors for SLAC collimators in units V/pC/mm.

| # | Geometry | Theory | Measured |
|---|---|---|---|
| 1 | Rect. | 2.5 | 1.24 ± 0.12 |
| 2 | Square | 1.4 | 1.38 ± 0.11 |
| 3 | Rect. | 2.5 | 1.38 ± 0.11 |
| 4 | Rect. | 0.6 | 0.54 ± 0.05 |

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