ANALYTICAL ESTIMATION OF THE BEAM-BEAM LIMITED DYNAMIC APERTURES AND LIFETIMES IN e^+e^- CIRCULAR COLLIDERS

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Abstract

Physically speaking, the delta function like beam-beam nonlinear forces at interaction points (IPs) act as a sum of delta function nonlinear multipoles. By applying the general theory established in ref. [1], in this paper we exame analytically the beam-beam interaction limited dynamic apertures and the corresponding beam lifetimes for both the round and the flat beams. Relations between the beam-beam limited beam lifetimes and the beam-beam tune shifts are established, which show clearly why experimentally one has always a maximum beam-beam tune shift, $\xi_{y,max}$, around 0.045 for e^+e^- circular colliders, and why one can use round beams to double this value approximately.

1 INTRODUCTION

Due to the importance of beam-beam effects, enormous efforts have been made toward better understandings [2]-[17]. Physically speaking, the delta function like beam-beam nonlinear forces at interaction points (IPs) act as a sum of delta function nonlinear multipoles. In ref. [1] we have established a general theory to study analytically in detail the delta function multipoles and their combined effects on the dynamic apertures in circular storage rings, and in this paper we will apply these general analytical formulae to the case of beam-beam interactions and find the corresponding beam dynamic apertures and beam lifetimes. Finally, we will show quantitatively why there exists a maximum beam-beam tune shift, $\xi_{y,max}$, around 0.045 for flat beams in e^+e^- circular colliders, and why this number can be almost doubled for round colliding beams.

2 BEAM-BEAM INTERACTIONS

For two head on colliding bunches, the incoherent kick felt by each particle can be calculated as [10]:

$$\delta y' + i\delta x' = -\frac{N_e r_e}{\gamma_*} f(x, y, \sigma_x, \sigma_y) \tag{1}$$

where x' and y' are the horizontal and vertical slopes, N_e is the particle population in the bunch, r_e is the electron classical radius (2.818×10^{-15} m), σ_x and σ_y are the standard deviations of the transverse charge density distribution of the counter-rotating bunch at IP, γ_* is the normalized particle's energy, and $*$ denotes the test particle and the bunch to which the test particle belongs. When the bunch is Gaussian $f(x, y, \sigma_x, \sigma_y)$ can be expressed by Basseti-Erskine formula [7]. For the round beam (RB) and the flat beam (FB) cases one has the incoherent beam-beam kicks expressed as [3][9][10]:

$$\delta r' = -\frac{2N_e r_e}{\gamma_* r}\left(1 - \exp\left(-\frac{r^2}{2\sigma^2}\right)\right) \qquad \text{(RB)} \tag{2}$$

$$\delta x' = -\frac{2\sqrt{2}N_e r_e}{\gamma_* \sigma_x}\exp\left(-\frac{x^2}{2\sigma_x^2}\right)\int_0^{\frac{x}{\sqrt{2}\sigma_x}} e^{u^2} du \qquad \text{(FB)} \tag{3}$$

$$\delta y' = -\frac{\sqrt{2\pi}N_e r_e}{\gamma_* \sigma_x}\exp\left(-\frac{x^2}{2\sigma_x^2}\right)\mathrm{erf}\left(\frac{y}{\sqrt{2}\sigma_y}\right) \qquad \text{(FB)} \tag{4}$$

where $r = \sqrt{x^2 + y^2}$. Now we want to calculate the average kick felt by the test particle since the probability to find the transverse displacement of the test particle is not constant (in fact, the probability function is the same as the charge distribution of the bunch to which the test particle belongs in lepton machines due to synchrotron radiations). In the following we assume that the transverse sizes for the two colliding bunches at IP are exactly the same. For the round beam case after averaging one gets[3][11]:

$$\delta \bar{r}' = -\frac{2N_e r_e}{\gamma_* \bar{r}}\left(1 - \exp\left(-\frac{\bar{r}^2}{4\sigma^2}\right)\right) \qquad \text{(RB)} \tag{5}$$

Although this expression is the same as that of the coherent beam-beam kick for round beams, one should keep in mind that we are not finding coherent beam-beam kick originally, and the difference will be obvious when we treat the vertical motion in the case of flat beams. For the flat beam case, we will treat the horizontal and vertical planes separately. As far as the horizontal kick is concerned, the horizontal kick depends only on one displacement variable just similar to the round beam case, we will use its coherent form expressed as follows [9][11]:

$$\delta x' = -\frac{2N_e r_e}{\gamma_* \sigma_x}\exp\left(-\frac{x^2}{4\sigma_x^2}\right)\int_0^{\frac{x}{2\sigma_x}} \exp(u^2) du \qquad \text{(FB)} \tag{6}$$

where σ_x in the incoherent formula in ref. [9] has been replaced by $\Sigma_x = \sqrt{2}\sigma_x$ (for two identical Gaussian colliding beams) according to Hirata theorem demonstrated in the appendix A of ref. [11]. As for the vertical kick, however, one has to make an average over eq. 4 with the horizontal probability distribution function of the test particle, and one gets [10]:

$$\delta y' = \frac{-\sqrt{2\pi}N_e r_e < \exp\left(-\frac{x^2}{2\sigma_x^2}\right) >_x \mathrm{erf}\left(\frac{y}{\sqrt{2}\sigma_y}\right)}{\gamma_* \sigma_x} \qquad \text{(FB)} \tag{7}$$

where $<>_x$ means the average over the horizontal probability distribution function of the test particle, and for two identical colliding Gaussian beams $<>_x = 1/\sqrt{2}$. It is obvious that eq. 7 is not the expression for the coherent beam-beam kick. To study both round and flat beam cases, we expand $\delta\bar{r}'$ at $x = 0$ (for round beam we study only vertical plane since the formalism in the horizontal plane is the same), $\delta x'$ and $\delta y'$ expressed in eqs. 5, 6 and 7, respectively, into Taylor series, and the differential equations of the motion of the test particle in the transverse planes can be obtained from the corresponding Hamiltonians expressed as:

$$H = \frac{p_y^2}{2} + \frac{K_y(s)}{2}y^2 + \frac{N_e r_e}{\gamma_*}\Big(\frac{1}{4\sigma^2}y^2 - \frac{1}{64\sigma^4}y^4 + \frac{1}{1152\sigma^6}y^6 - \frac{1}{24576\sigma^8}y^8 + \cdots\Big)\sum_{k=-\infty}^{\infty}\delta(s-kL) \qquad \text{(RB)} \qquad (8)$$

$$H_x = \frac{p_x^2}{2} + \frac{K_x(s)}{2}x^2 + \frac{N_e r_e}{2\gamma_*}\Big(\frac{1}{\sigma_x^2}x^2 - \frac{1}{12\sigma_x^4}x^4 + \frac{1}{180\sigma_x^6}x^6 - \frac{1}{3360\sigma_x^8}x^8 + \cdots\Big)\sum_{k=-\infty}^{\infty}\delta(s-kL) \qquad \text{(FB)} \qquad (9)$$

$$H_y = \frac{p_y^2}{2} + \frac{K_y(s)}{2}y^2 + \frac{N_e r_e}{\sqrt{2}\gamma_*}\Big(\frac{1}{\sigma_x\sigma_y}y^2 - \frac{1}{12\sigma_x\sigma_y^3}y^4 + \frac{1}{120\sigma_x\sigma_y^5}y^6 - \frac{1}{1344\sigma_x\sigma_y^7}y^8 + \cdots\Big)\sum_{k=-\infty}^{\infty}\delta(s-kL) \qquad \text{(FB)} \qquad (10)$$

where $p_x = dx/ds$ and $p_y = dy/ds$.

3 REVIEW

In ref. [1] we have studied analytically the one dimensional ($y = 0$) dynamic aperture of a storage ring described by the following Hamiltonian:

$$H = \frac{p^2}{2} + \frac{K(s)}{2}x^2 + \frac{1}{3!B\rho}\frac{\partial^2 B_z}{\partial x^2}x^3 L\sum_{k=-\infty}^{\infty}\delta(s-kL) + \frac{1}{4!B\rho}\frac{\partial^3 B_z}{\partial x^3}x^4 L\sum_{k=-\infty}^{\infty}\delta(s-kL) + \cdots \qquad (11)$$

where

$$B_z = B_0(1+xb_1+x^2b_2+x^3b_3+x^4b_4+\cdots+x^{m-1}b_{m-1}+\cdots) \qquad (12)$$

The dynamic aperture corresponding to each multipole is given as:

$$A_{dyna,2m,x}(s) = \sqrt{2\beta_x(s)}\left(\frac{1}{m\beta_x^m(s_{2m})}\right)^{\frac{1}{2(m-2)}} \times \left(\frac{\rho}{|b_{m-1}|L}\right)^{1/(m-2)} \qquad (13)$$

where s_{2m} is the location of the $2m$th multipole, $\beta_x(s)$ is the beta function in x plane. Since these results are general, we have tried to avoid to assign the freedom of motion, x, a specific name, such as horizontal, or vertical plane.

m	4	6	8	10	12
$C_{m,RB}$	16	192	3072	61440	1474560
$C_{m,FB,x}$	3	30	420	7560	166320
$C_{m,FB,y}$	3	20	168	1728	21120

Table 1: summary of multipole coefficients

4 BEAM-BEAM LIMITED DYNAMIC APERTURES

To make use of the general dynamic aperture formulae recalled in section 3, one needs only to find the equivalence relations by comparing three Hamiltonians expressed in eqs. 8, 9, and 10 with eq. 11, and it is found by analogy that:

$$\frac{b_{m-1}}{\rho}L = \frac{N_e r_e}{C_{m,RB}\gamma_*\sigma^m} \qquad \text{(RB)} \qquad (14)$$

$$\frac{b_{m-1}}{\rho}L = \frac{N_e r_e}{C_{m,FB,x}2\gamma_*\sigma_x^m} \qquad (\text{FB}, x) \qquad (15)$$

$$\frac{b_{m-1}}{\rho}L = \frac{N_e r_e}{C_{m,FB,y}\sqrt{2}\gamma_*\sigma_x\sigma_y^{m-1}} \qquad (\text{FB}, y) \qquad (16)$$

where $C_{m,RB}$, $C_{m,FB,x}$, and $C_{m,FB,y}$ are given in Table 1. Now by inserting eqs. 14-16 into eq. 13 one can calculate dynamic apertures of different multipoles due to nonlinear beam-beam forces. Given the dynamic aperture of the ring without the beam-beam effect as $A_{x,y}$, the total dynamic aperture including the beam-beam effect can be estimated usually as:

$$A_{total,x,y}(s) = \frac{1}{\sqrt{\frac{1}{A_{x,y}(s)^2} + \frac{1}{A_{bb,x,y}(s)^2}}} \qquad (17)$$

In the following we will consider the case of $A_{total,x,y}(s) \approx A_{bb,x,y}(s)$, and we find:

$$\mathcal{R}_{y,8} = \frac{A_{dyna,8,y}(s)}{\sigma_*(s)} = \left(\frac{16\gamma_*\sigma^2}{N_e r_e\beta_y(s_{IP})}\right)^{1/2} \qquad \text{(RB)} \qquad (18)$$

$$\mathcal{R}_{x,8} = \frac{A_{dyna,8,x}(s)}{\sigma_{*,x}(s)} = \left(\frac{6\gamma_*\sigma_x^2}{N_e r_e\beta_x(s_{IP})}\right)^{1/2} \qquad \text{(FB)} \qquad (19)$$

$$\mathcal{R}_{y,8} = \frac{A_{dyna,8,y}(s)}{\sigma_{*,y}(s)} = \left(\frac{3\sqrt{2}\gamma_*\sigma_x\sigma_y}{N_e r_e\beta_y(s_{IP})}\right)^{1/2} \qquad \text{(FB)} \qquad (20)$$

Recalling and using the definitions of the beam-beam tune shifts ξ_x and ξ_y, one can simplify the above defined normalized dynamic apertures. As general results one finds:

$$\mathcal{R}_{y,2m} = \frac{A_{dyna,2m,y}(s)}{\sigma_{*,y}(s)} = \left(\frac{2^{\frac{m-2}{2}}C_{m,RB}}{4\pi\sqrt{m}\xi_y^*}\right)^{\frac{1}{m-2}} \qquad \text{(RB)} \qquad (21)$$

$$\mathcal{R}_{x,2m} = \frac{A_{dyna,2m,x}(s)}{\sigma_{*,x}(s)} = \left(\frac{2^{\frac{m-2}{2}}C_{m,FB,x}}{2\sqrt{m}\pi\xi_x^*}\right)^{\frac{1}{m-2}} \qquad \text{(FB)} \qquad (22)$$

$$\mathcal{R}_{y,2m} = \frac{A_{dyna,2m,y}(s)}{\sigma_{*,y}(s)}$$
$$= \left(\frac{2^{\frac{m-2}{2}} C_{m,FB,y}}{\sqrt{2m}\pi\xi_y^*} \right)^{\frac{1}{m-2}} \qquad \text{(FB)} \qquad (23)$$

5 BEAM-BEAM LIFETIMES

We take the beam-beam limited dynamic aperture as the rigid mechanical boundary. Based on this physical point of view we can use the beam quantum lifetime formula [18] to estimate the beam lifetime due to beam-beam effect:

$$\tau_{bb} = \frac{\tau_y}{2}\left(\frac{<y^2>}{y_{max}^2}\right)\exp\left(\frac{y_{max}^2}{<y^2>}\right)$$
$$= \frac{\tau_y}{2}\left(\frac{\sigma_y(s)^2}{A_{dyna,y}(s)^2}\right)\exp\left(\frac{A_{dyna,y}(s)^2}{\sigma_y(s)^2}\right) \qquad (24)$$

where y_{max} is the boundary dimension, $<>$ denotes the average over the particle distribution, and τ_y is the synchrotron radiation damping time in vertical plane. In eq. 24 we have replaced the y_{max} and $<y^2>$ by $A_{dyna,y}$ and σ_y^2, respectively. When the beam-beam octupole nonlinear force dominates the dynamic aperture, by inserting eqs. 18, 18, and 20 into eq. 24, or inserting eqs. 21, 22, and 23 into eq. 24 one gets:

$$\tau_{bb,y}^* = \frac{\tau_y^*}{2}\left(\frac{4}{\pi\xi_y^*}\right)^{-1}\exp\left(\frac{4}{\pi\xi_y^*}\right) \qquad \text{(RB)} \qquad (25)$$

$$\tau_{bb,x}^* = \frac{\tau_x^*}{2}\left(\frac{3}{\pi\xi_x^*}\right)^{-1}\exp\left(\frac{3}{\pi\xi_x^*}\right) \qquad \text{(FB)} \qquad (26)$$

$$\tau_{bb,y}^* = \frac{\tau_y^*}{2}\left(\frac{3}{\sqrt{2}\pi\xi_y^*}\right)^{-1}\exp\left(\frac{3}{\sqrt{2}\pi\xi_y^*}\right) \qquad \text{(FB)} \qquad (27)$$

6 THE MAXIMUM BEAM-BEAM TUNE SHIFTS

Now it is high time for us to discuss the maximum beam-beam tune shift problem. In literatures the term “maximum beam-beam tune shift” of a specific machine is not well defined. One of the reasonable definitions would be that the maximum beam-beam tune shift corresponding to a well defined minimum beam-beam limited lifetime. In this paper we propose to take this well defined minimum beam-beam limited lifetime as one hour (the idea is to reduce eq. 17 to $A_{total}(s) \approx A_{bb}(s)$, and to have a machine still working!). Assuming that for both round and flat beam cases one has the same τ_y, from eqs. 25, 26 and 27 one finds the following relations:

$$\xi_{y,max}^{RB} = \frac{4\sqrt{2}}{3}\xi_{y,max}^{FB} = 1.89\xi_{y,max}^{FB} \qquad (28)$$

and

$$\xi_{x,max}^{FB} = \sqrt{2}\xi_{y,max}^{FB} \qquad (29)$$

It is proved theoretically why round beam scheme can almost double the $\xi_{y,max}$ of flat beam scheme as previously discovered in the numerical simulations [13][14], and why the vertical beam-beam tune shift reaches its limit earlier than the horizontal one. Quantitatively, taking $\tau_y = 30$ ms, one finds that $\xi_{y,max,FB}(\tau_{bb} = 1$ hour$)= 0.0447$, $\xi_{x,max,FB}(\tau_{bb} = 1$ hour$)= 0.0632$, and $\xi_{y,max,RB}(\tau_{bb} = 1$ hour$)= 0.0843$. The detailed discussion on how to choose working point and the effects of a finite crossing angle can be found in ref. [17].

7 CONCLUSION

In this paper we have established analytical formulae for the beam-beam interaction limited dynamic apertures and beam lifetimes in e^+e^- circular colliders for both round and flat beam cases. It is shown analytically why for flat colliding beams one has always $\xi_{y,max}$ around 0.045 and why this value can be almost doubled by using round beams.

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