# ORDER-BY-ORDER SYMPLECTIFICATION OF TRUNCATED LIE MAPS

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*Abstract*

It is known that the symplecticity property for Hamiltonian systems is lost for truncated Lie maps. In the case of long time evolution this fact can lead to spurious effects appearance and/or to real effects vanishing. In this report an order-by-order symplectification method for truncated Lie maps is described. This method is based on the matrix formalism for Lie algebraic tools. According to this formalism truncated Lie map presented as a set of two-dimensional matrices corresponded to nonlinear aberrations up to N order. Matrix elements can be evaluated using computer algebra codes and Kronecker sum and production tools. These block-matrices for accelerator lattices are elements of a corresponding database. The additional conservative conditions (in our case it is symplectic conditions) lead to linear homogeneous algebraic equations for matrix elements. Choosing basis elements (calculated using the matrix formalism algorithms) one can calculate the others. Resulting block-matrices guarantee the symplecticity of the truncated Lie map up to the N-th order. These linear relations can be calculated in advance and stored in a symbolic database. Finally, this method is applied to some practical problems of particle physics.


## 1 INTRODUCTION

It is known that the most part of beam physics problems can be described using the Hamiltonian presentation. Main properties of similar systems are in common practice of qualitative investigations. But the practical calculations based only on numerical algorithms do not guarantee in general the symplecticity property which is inherent to all Hamiltonian systems. Failing of this property can produce losing of real effects and to acquisition of false effects. That is why all commonly used numerical methods should have the symplecticity property. In that case the simulation process will guarantee adequate and accurate results. In recent years there have been successful results in computer modeling of long beam evolution using the Lie algebraic methods. According to this approach a simulator constructs high-order maps and use them for the design, optimization and operation of beamlines. However the practical realization of this powerful approach does not guarantee the symplecticity property automatically. Indeed the realization of Lie methods usually uses truncated series in different forms. As it is known these truncated series have not properties intrinsical to the starting map. There are some works (see, e.g. [1]–[3]) where some symplectification algorithms are described. These algorithms have numerical character and so have all imperfections residing to all numerical methods and algorithms.

In this report a new approach for step-by-step symplectification for the Lie algebraic methods are suggested. In contrast to usual numerical approaches the way uses matrix formalism for the Lie algebraic tools in symbolic mode. This allows us to create very simple correction formulae which guarantee the symplecticity properties in all orders up to some hand-picked approximation order. These formulae follow from linear algebraic equations which can be solved in symbolic form easily. It should be noted two main advantages of symbolic mode: the first of them is flexibility and the second is a computational accuracy in high orders. More over, all necessary computations (having been done in advance) can be stored in a special database and used in necessity.

## 2 MATHEMATICAL BACKGROUNDS

The Lie algebraic tools are usually used in so the called polynomial presentation (see, e.g. [4]–[5]). In this presentation the starting system Hamiltonian is expanded into a sequence of homogeneous (over the phase vector $X$) polynomials. Further this polynomial presentation is applied for tensor presentation creation of designed high-order maps. According to the matrix formalism for Lie algebraic methods (see, e.g. [6]) the high-order map is presented in the form of a block-matrix, every block of which is a two-dimensional matrix calculated in a symbolic form in advance.

### 2.1 General Layout of the Matrix Formalism for Lie Methods

The motion equation for beam particles can be written in the following general form:

$$\frac{d\mathbf{X}}{ds} = \mathbf{F}(\mathbf{X}, \mathbf{U}, \mathbf{B}, s), \tag{1}$$

$\mathbf{X}$ is a phase vector, $\mathbf{U} = \mathbf{U}(t)$ – a vector of control functions, $\mathbf{B}$ – a vector of control parameter $s$– an independent variable (e.g. the length of a reference orbit). For the initial conditions one can write

$$\mathbf{X}(s_0) = \mathbf{X}_0, \quad s \in [s_0, \mathrm{S}], \quad \mathbf{X}_0 \in \mathfrak{M}_0, \tag{2}$$

where $\mathfrak{M}_0$ is an initial phase beam portrait, S – is a characteristic length (e.g. the accelerator period). The right side of the Eq. 2 can be written in the form of the following series:

$$\frac{d\mathbf{X}}{ds} = \sum_{k=0}^{\infty} \mathbb{P}^{1k}(\mathbf{U}, \mathbf{B}, s)\mathbf{X}^{[k]},$$

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where $\mathbf{X}^{[k]}$ is so called Kronecker power of the $k$-th order: $\mathbf{X}^{[k]} = \underbrace{\mathbf{X} \otimes \ldots \otimes \mathbf{X}}_{\text{k times}}$. The matrices $\mathbb{P}^{1k}$ can be easily calculated using computer algebra codes (e.g. Maple V) for different control elements. This matrices play a role of bricks during the computer models construction. For solving of the Eq. 3 the Lie algebraic methods are used according to them one can write

$$\mathbf{X}(\mathbf{X}_0, \mathbf{U}, \mathbf{B}, s) = \sum_{k=0}^{\infty} \mathbb{M}^{1k}(\mathbf{U}, \mathbf{B}, s)\mathbf{X}_0^{[k]}. \tag{3}$$

The matrices $\mathbb{M}^{1k}$ can be presented in the form

$$\mathbb{M}^{1k} = \mathbb{M}^{11}\mathbb{Q}^{1k},$$

where $\mathbb{Q}^{1k}$ are new matrices and $\mathbb{M}^{11}$ is the linear transfer matrix generated by the Lie map $\mathcal{M}$, defined as the solution of linear motion equations

$$\frac{d\mathbf{X}}{ds} = \mathbb{P}^{11}(\mathbf{U}, \mathbf{B}, s)\mathbf{X}. \tag{4}$$

The matrices $\mathbb{Q}^{1k}$ can be calculated using the algorithms of the matrix formalism.

Knowledge of symbolic presentation for $\mathbb{P}^{1k}$ permits to compute the matrices $\mathbb{M}^{1k}$ or the matrices $\mathbb{Q}^{1k}$ too. It should be noted that knowledge of the symbolic presentation of $\mathbb{M}^{1k}$, $\forall k \leq N$ (here $N$ is an approximation order) allows also to solve different problems of modeling and optimization beamlines. Starting from the matrix formalism, one can solve different problems (for example, of symmetry and invariants searching [7], synthesis of beamlines with desired characteristics [8], the problem of space charge forces [9]). It is obvious that in practice the expansion (3) should be truncated at some approximation order $N$. But in this case the property of symplecticity is lost.

### 2.2 Symplectic Conditions for Block Matrices

In this paper, a new method to correct block matrices $\mathbb{M}^{1k}$ step-by-step is presented. The Jacobi matrix $\mathbb{M}(\mathbf{X}) = \mathbb{M}(\mathbf{X}; s \,|\, s_0)$ for our Lie map $\mathcal{M}$:

$$\mathbb{M}(\mathbf{X}) = \partial(\mathcal{M} \cdot \mathbf{X})/\partial \mathbf{X}^{\mathrm{T}}.$$

Here and in the following the dependence on the vectors $\mathbf{U}, \mathbf{B}$ and variable $s$ is omitted. The starting point of the correction algorithms is the symplecticity condition for the Jacobi matrix $\mathbb{M}$:

$$\mathbb{M}(\mathbf{X})\mathbb{J}\mathbb{M}^{\mathrm{T}}(\mathbf{X}) = \mathbb{J},$$

where $\mathbb{J}$ is a canonical symplectic matrix,

$$\mathbb{J} = \begin{pmatrix} \mathbb{O} & \mathbb{E} \\ -\mathbb{E} & \mathbb{O} \end{pmatrix}.$$

Here $\mathbb{O}$ and $\mathbb{E}$ are zero matrix and unit matrix correspondingly. Using the matrix presentation for Lie maps the Jacobi matrix $\mathbb{M}(\mathbf{X})$ can be written in the form

$$\mathbb{M}(\mathbf{X}) = \sum_{k=1}^{\infty} \mathbb{M}^{1k} \frac{\partial \mathbf{X}^{[k]}}{\partial \mathbf{X}^*}.$$

The properties of the Kronecker sum and product allows to evaluate

$$\mathbb{M}(\mathbf{X}) = \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} \mathbb{M}^{1k}\mathbf{X}^{[j]} \otimes \mathbf{E} \otimes \mathbf{X}^{[k-j-1]},$$

or

$$\underbrace{\left(\mathbb{M}^{11}\right)^* \mathbb{J}_0\mathbb{M}^{11}}_{k=l=1} + \underbrace{\left(\mathbf{X} \otimes \mathbb{E} + \mathbb{E} \otimes \mathbf{X}\right)^* \left(\mathbb{M}^{12}\right)^* \mathbb{J}_0\mathbb{M}^{11}}_{k=2,\ l=1} +$$
$$+ \underbrace{\left(\mathbb{M}^{11}\right)^* \mathbb{J}_0\mathbb{M}^{12} \left(\mathbf{X} \otimes \mathbb{E} + \mathbb{E} \otimes \mathbf{X}\right)}_{k=1,\ l=2} +$$
$$+ \sum_{\substack{k,l=1 \\ k+l>3}}^{\infty} \left(\mathbf{X}^{\odot k}\right)^* \left(\mathbb{M}^{1k}\right)^* \mathbb{J}_0\mathbf{X}^{\odot l} = \mathbb{J}_0, \tag{5}$$

where $\otimes$ is the Kronecker multiplication, and $\odot$ is a new operation defined according to the following rule

$$\mathbf{X}^{\odot(k-1)} = \sum_{j=0}^{k-1} \mathbf{X}^{[j]} \otimes \mathbb{E} \otimes \mathbf{X}^{[k-j-1]}.$$

As the equation (5) should be fulfilled for all phase vectors $\mathbf{X}$ one can write the following sequence of the equations:

$$\left(\mathbb{M}^{11}\right)^* \mathbb{J}_0\mathbb{M}^{11} = \mathbb{J}_0, \tag{6}$$

$$\left(\mathbf{X}^{\odot 1}\right)^* \left(\mathbb{M}^{12}\right)^* \mathbb{J}_0\mathbb{M}^{11} + \left(\mathbb{M}^{11}\right)^* \mathbb{J}_0\mathbb{M}^{12}\mathbf{X}^{\odot 1} +$$
$$+ \sum_{k,l \geq 2} \left(\mathbf{X}^{\odot k}\right)^* \left(\mathbb{M}^{1k}\right)^* \mathbb{J}_0\mathbb{M}^{1l}\mathbf{X}^{\odot l} = 0. \tag{7}$$

These equations are required symplecticity conditions of truncated map in the matrix presentation.

## 3 SOLUTION OF THE SYMPLECTIC CONDITIONS

### 3.1 The Basic Equations and Definitions

As it is known the matricant of the Eq. (4) can be presented in the following form of the following operator exponent

$$\mathbb{M}^{11}(s \,|\, s_0; \mathbb{P}^{11}) = \exp\left(\mathbb{J}\tilde{\mathbb{P}}^{11}\left(s \,|\, s_0; \mathbb{P}^{11}\right)\right),$$

where $\tilde{\mathbb{P}}^{11}$ is a symmetric matrix dependent on the "old" matricant $\mathbb{P}^{11}$. It can be calculated with the help of lie algebraic tools. In particular, in the quasistationary case (when the commutator is equal to zero: $\{\int_{s_0}^{s} \mathbb{P}^{11}(\tau)d\tau, \mathbb{P}^{11}(\tau')\} = 0$) we have

$$\tilde{\mathbb{P}}^{11}(s \,|\, s_0; \mathbb{P}^{11}) = \int_{s_0}^{s} \mathbb{P}^{11}(\tau)d\tau.$$

In the arbitrary case the matrix $\tilde{\mathbb{P}}^{11}$ can be calculated with the help of the Magnus presentation. It is not difficult to demonstrate that this matrix will be symmetric in any case. The symplecticity of the matrix $\mathbb{M}^{11}$ follows from the theory of symplectic matrices and the Lie brackets properties. The Eq. (7) generate the infinity chain of algebraic equations for block matrices $\mathbb{M}^{1k}$. It is not difficult to show that the conditions for the matrix $\mathbb{M}^{12}$ are not depend on elements of the matrices $\mathbb{M}^{1k}$ for $k > 2$. The rest of matrices are fulfilled to the infinity chain of coupled equations.

Using the following presentation $\mathbb{M}^{1k} = \mathbb{M}^{11}\mathbb{Q}^{1k}$ and $\mathbb{Q}^{11} = \mathbb{E}$ the Eq. (7) can be rewritten in the more simple form

$$\sum_{k,l\geq 0}^{\infty} \left(\mathbf{X}^{\odot k}\right)^* \left(\mathbb{Q}^{1\,(k+1)}\right)^* \mathbb{J}_0 \mathbb{Q}^{1\,(l+1)} \mathbf{X}^{\odot l} = 0. \tag{8}$$

Both the Eq. (7) and the Eq. (8) are decomposed on the following chain of the matrix equations:

$$\sum_{\substack{k+l=m\\ m\geq 1}} \left(\mathbf{X}^{\odot k}\right)^* \left(\mathbb{Q}^{1\,(k+1)}\right)^* \mathbb{J}_0 \mathbb{Q}^{1\,(l+1)} \mathbf{X}^{\odot l} = 0. \tag{9}$$

The elements of the matrices $\mathbb{Q}^{1j}$ are the homogeneous polynomials over components of the phase vector $\mathbf{X}$. The equations for these elements can be presented in the form of linear algebraic equations, which can be solved using computer algebra codes easily. Similar solutions can be stored in a special data base and used as required.

### 3.2 The Example of the Corrected Matrices

As a demonstration example let consider the first equality (9) (in the case of $n = 1$: $\mathbf{X} = (x, p_x)^{\mathrm{T}}$)

$$\begin{aligned}(\mathbf{X}\otimes\mathbb{E} + \mathbb{E}\otimes\mathbf{X})^* \left(\mathbb{Q}^{12}\right)^* \mathbb{J}_0 +\\ + \mathbb{J}_0 \mathbb{Q}^{12} (\mathbf{X}\otimes\mathbb{E} + \mathbb{E}\otimes\mathbf{X}) = 0.\end{aligned}$$

Denoting $\{\mathbb{Q}^{12}\}_{ik} = q_{ik}$ one can evaluate

$$\begin{pmatrix} 0 & \substack{(2q_{11}+q_{22})x+\\ (q_{12}+2q_{23})p_x} \\ \substack{-(2q_{11}+q_{22})x-\\ (q_{12}+2q_{23})p_x} & 0 \end{pmatrix} = 0.$$

Thus in our case we obtain a single scalar equation

$$(2q_{11} + q_{22})x + (2q_{23} + q_{12})p_x = 0,$$

or since that $x$, $p_x$ are arbitrary we have:

$$2q_{11} + q_{22} = 0, \quad q_{12} + 2q_{23} = 0. \tag{10}$$

The equations (10) can be resolved for example with regard to $q_{22}$ and $q_{23}$: $q_{22} = -2q_{11}$, $q_{23} = -q_{11}/2$. So the symplecticity conditions for the second order will be fulfilled automatically, if the matrix $\mathbb{M}^{12}$ has the form

$$\mathbb{M}^{12} = \mathbb{M}^{11} \cdot \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & -2q_{11} & -\frac{1}{2}q_{12} \end{pmatrix},$$

where $q_{ik}$ are calculated according to the matrix formalism [10]. It should be noted that these conditions are much simpler of similar conditions for the elements of the matrix $\mathbb{M}^{12}$. Indeed the corresponding conditions for elements of the matrix $\mathbb{M}^{12}$ have the following form:

$$\begin{aligned} r_{11}m_{22} - r_{21}m_{12} + 2(r_{22}m_{11} - r_{12}m_{21}) &= 0,\\ 2(r_{11}m_{23} - r_{21}m_{13}) + r_{22}m_{12} - r_{12}m_{22} &= 0, \end{aligned} \tag{11}$$

where $r_{ik}$ and $m_{ik}$ are elements of the matrices $\mathbb{M}^{11}$ and $\mathbb{M}^{12}$ correspondingly. As one can see from the (14) these equations connects all elements of the matrix $\mathbb{M}^{12}$. Solution of the Eqs. (11) demands more logical difficulties, because similar equations confuse all elements of the matrix $\mathbb{M}^{12}$. Up to table of symbols the Eqs. (11) coincide with corresponding conditions in [10].

This approach was realized for some practical problems of long time evolution of particle beams in cyclic accelerators (including space-charge forces). Comparison of computer experiments results demonstrates the necessary stability of this algorithm.

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