# ANALYTICAL FORMULAE FOR THE WAKEFIELDS PRODUCED BY THE NONRELATIVISTIC CHARGED PARTICULES IN PERIODIC DISK-LOADED STRUCTURES

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*Abstract*

In this paper we consider the wakefields induced in periodic disk-loaded cavities by charged particles of their velocities less than that of light. In frequency domain the particle velocity dependent wakefields can be calculated by generalizing the analytical formulae given in ref. 1. The physical picture of this effect can be drawn as that the frequencies of the excited modes in cavities felt by the particles are increased by a factor of $1/\beta$ ($\beta$ is the normalized particle's velocity). Some examples are given to demonstrate the utility of these formulae in obtaining the quantities, such as loss factors, short range and long range wakefields as functions of cavity dimension, bunch length, and particle velocity.

## 1 INTRODUCTION

The wakefields produced by nonrelativistic charged particles in accelerating structures is becoming a subject of research together with the studies of high power proton (or $H^-$) linear accelerators, such as the facility so-called Accelerator Production of Tritium (APT) [2]. Since the linac in the high energy part is superconducting type, one is interested in the energy deposited inside the cavities by the passing particles [3], and also in the wakefields induced instabilities in the accelerating structures, such as in the ionization cooling channel of a muon collider [4]. Compared with the wakefields produced by the highly relativistic ($\beta = v/c = 1$) charged particles, the wakefields corresponding to the nonrelativistic particles are velocity dependent. Restricted to the problem of particle cavity interactions, the physical picture of this effect can be drawn as that the frequencies of the excited modes felt by the particles are increased by a factor of $1/\beta$. Having this physical picture in mind, one can generalize the formalism of the analytical wakefield calculation for cavities in time domain with $\beta = 1$ to the general case of $\beta \leq 1$ in a rather straight forward way. In the following sections we will give a set of analytical formulae to calculate wakefields and some examples to demonstrate their practical applications.

## 2 THEORY

We restrict ourselves to the particle-cavity interactions and treat the wakefield problem in frequence domain. In this section we generalize the analytical formulae derived in ref. 1 to calculate the wakefields produced by the particles with their velocities less than that of light in a disk-loaded structure as shown in Fig. 1. The delta wakefield functions of a point charge traversing a disk-loaded structure can be calculated by using the following formulae:

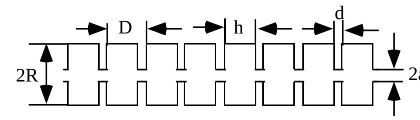


Figure 1: Disk-loaded accelerating structure

$$W_z(\tau) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{l=0}^{\infty} W_{z,mnl}(\tau) \tag{1}$$

$$W_r(\tau) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{l=0}^{\infty} W_{r,mnl}(\tau) \tag{2}$$

$$W_\phi(\tau) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{l=0}^{\infty} W_{\phi,mnl}(\tau) \tag{3}$$

where

$$W_{z,mnl}(\tau) = 2k_{mnl}\left(\frac{r}{a}\right)^m\left(\frac{r_q}{a}\right)^m \cos(m\phi)\cos(\omega_{mnl}\tau) \tag{4}$$

$$W_{r,mnl}(\tau) = 2m\frac{ck_{mnl}}{\omega_{mnl}a}\left(\frac{r}{a}\right)^{m-1}\left(\frac{r_q}{a}\right)^m \times \cos(m\phi)\sin(\omega_{mnl}\tau) \tag{5}$$

$$W_{\phi,mnl}(\tau) = -2m\frac{ck_{mnl}}{\omega_{mnl}a}\left(\frac{r}{a}\right)^{m-1}\left(\frac{r_q}{a}\right)^m \times \sin(m\phi)\sin(\omega_{mnl}\tau) \tag{6}$$

$$\omega_{mnl}^2 = c^2\left(\left(\frac{u_{mn}}{R}\right)^2 + \left(\frac{l\pi}{h}\right)^2\right) \tag{7}$$

where $\tau = \frac{s}{\beta c}$, $s$ is the distance between the exciting charge and a test charge, and $r_q$ is the transverse coordinate of the exciting charge. For a Gaussian bunch of charge $q$ one can calculate the integrated wakefield started from delta wakefield functions:

$$W_{G,z}(\tau) = \int_{-\infty}^{\tau} W_z(\tau - t)I(t)dt \tag{8}$$

$$W_{G,r}(\tau) = \int_{-\infty}^{\tau} W_r(\tau - t)I(t)dt \tag{9}$$

 

$$W_{G,\phi}(\tau) = \int_{-\infty}^{\tau} W_{\phi}(\tau - t) I(t) dt \qquad (10)$$

$$I(t) = \frac{q}{(2\pi)^{1/2}\sigma_t} \exp\left(-\frac{t^2}{2\sigma_t^2}\right) \qquad (11)$$

where $\sigma_t = \frac{\sigma_z}{\beta c}$. If $\tau \geq 3\sigma_t$ eqs. 8, 9 and 10 can be replaced by the following expressions:

$$W_{G,z}(\tau) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{l=0}^{\infty} W_{z,mnl}(\tau) \exp(-\frac{\omega_{mnl}^2 \sigma_t^2}{2}) \qquad (12)$$

$$W_{G,r}(\tau) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{l=0}^{\infty} W_{r,mnl}(\tau) \exp(-\frac{\omega_{mnl}^2 \sigma_t^2}{2}) \qquad (13)$$

$$W_{G,\phi}(\tau) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{l=0}^{\infty} W_{\phi,mnl}(\tau) \exp(-\frac{\omega_{mnl}^2 \sigma_t^2}{2}) \qquad (14)$$

For the $m$th mode the total loss factor of a Gaussian bunch will be

$$K_m(\sigma_t) = \sum_{n=1}^{\infty}\sum_{l=0}^{\infty} k_{mnl}(\sigma_t)$$

$$= \sum_{n=1}^{\infty}\sum_{l=0}^{\infty} k_{mnl} \exp(-\omega_{mnl}^2 \sigma_t^2) \qquad (15)$$

The general expression of the loss factor $k_{mnl}$ corresponding to the $mnl$th passband [1] is generalized as:

$$k_{mnl} = \frac{2\xi h u_{mn}^2 J_m^2\left(\frac{u_{mn}}{R}a\right)}{\left(\left(\frac{u_{mn}}{R}\right)^2 + \left(\frac{l\pi}{h}\right)^2\right)\epsilon_0 D\pi R^4 J_{m+1}^2(u_{mn})} \times \left(\frac{S(x_1)^2 + S(x_2)^2}{4}\right) \qquad (16)$$

where

$$\xi = \begin{cases} 1, m \neq 0 \\ 1/2, m = 0 \end{cases} \qquad (17)$$

$$S(x) = \frac{\sin(x)}{x} \qquad (18)$$

and

$$x_1 = \frac{h}{2\beta}\left(\left(\left(\frac{u_{mn}}{R}\right)^2 + \left(\frac{l\pi}{h}\right)^2\right)^{1/2} - \frac{l\pi}{h}\right) \qquad (19)$$

$$x_2 = \frac{h}{2\beta}\left(\left(\left(\frac{u_{mn}}{R}\right)^2 + \left(\frac{l\pi}{h}\right)^2\right)^{1/2} + \frac{l\pi}{h}\right) \qquad (20)$$

To summarize, one finds that particle velocity $\beta$ has been taken into account through $\tau = \frac{s}{\beta c}$, $\sigma_t = \frac{\sigma_z}{\beta c}$, and in eqs. 19 and 20.

When $\beta = 1$, by setting $m = 0$, $n = 1$, and $l = 0$, one gets from eq. 16 the point charge fundamental mode loss factor of a disk-loaded structure as obtained before in ref. 5:

$$k_{010} = \frac{2J_0^2\left(\frac{u_{01}}{R}a\right)\sin^2\left(\frac{u_{01}h}{2R}\right)}{\epsilon_0 \pi h D J_1^2(u_{01}) u_{01}^2} \qquad (21)$$

Obviously when $a = 0$ and $h = D$, eq. 21 gives the point charge fundamental mode loss factor of a closed pill-box cavity, and when $a = R$ one gets $k_{mnl} \equiv 0$, which corresponds to a round beam pipe without resistive losses.

What should be kept in mind is that in this paper we consider the wakefields in stead of wake potential, and that the wakefields and loss factors have units of V/C/m.

## 3 EXAMPLES

Equipped with the general analytical formulae derived above, let's look at a SLAC-type periodic disk-loaded structure with $a = 1$ cm, $R = 4.02$ cm, $h = 2.92$ cm, and $D = 3.5$ cm with $\sigma_z = 1$ cm. Figs. 2 to 5 shows the differences of the loss factors, short range longitudinal and diople mode wakefields, and long range longitudinal wakefields at the two different particle velocities. More comparison results can be found in ref. [6].

## 4 CONCLUSION

In this paper we have generalized the analytical formulae of the wakefields in a periodic disk-loaded structure (including the closed pill-box cavity) in ref. 1 to the case where the charged particle's velocity can be less than that of light. The advantages of these formulae are that they take the beam pipe radius into account, that it is very convenient for them to be included into the cavity design automation program, such as that under development in Los Alamos [7] to calculate the particle velocity dependent wakefields and loss factors in accelerating structures, such as APT-type structures [8], that they are very efficient to calculate the wakefields of very short bunch length [9], and that they can be even used to estimate the wakefields due to beam pipe surface roughness [10].

## 5 REFERENCES

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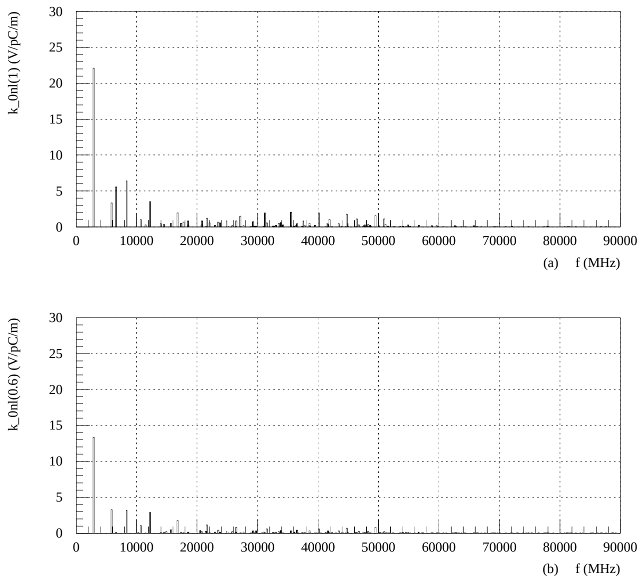


Figure 2: A periodic disk-loaded structure: $a = 0.01$ m, $h = 0.0292$ m, $D = 0.035$ m, and $\sigma_z = 0.01$ m. The monopole mode loss factors versus the frequency at (a) $\beta = 1$, and (b) $\beta = 0.6$.

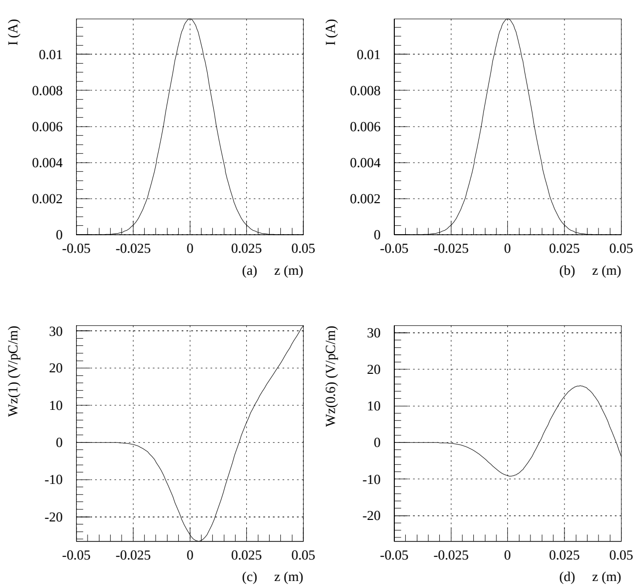


Figure 3: A periodic disk-loaded structure: $a = 0.01$ m, $h = 0.0292$ m, $D = 0.035$ m, and $\sigma_z = 0.01$ m. (a) and (b) are the Gaussian bunch current distributions of a total charge of 1 pC. The short range longitudinal wakefields at (c) $\beta = 1$, and (d) $\beta = 0.6$.

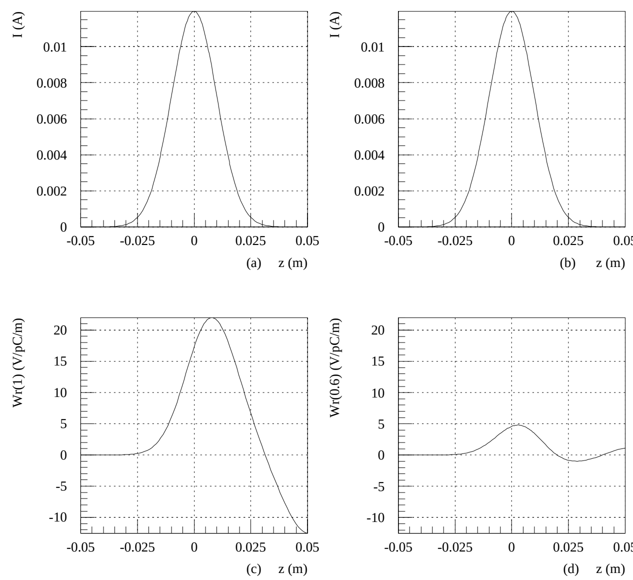


Figure 4: A periodic disk-loaded structure: $a = 0.01$ m, $h = 0.0292$ m, $D = 0.035$ m, and $\sigma_z = 0.01$ m. (a) and (b) are the Gaussian bunch current distributions of a total charge of 1 pC. The short range dipole mode wakefields ($r = a$) at (c) $\beta = 1$, and (d) $\beta = 0.6$.

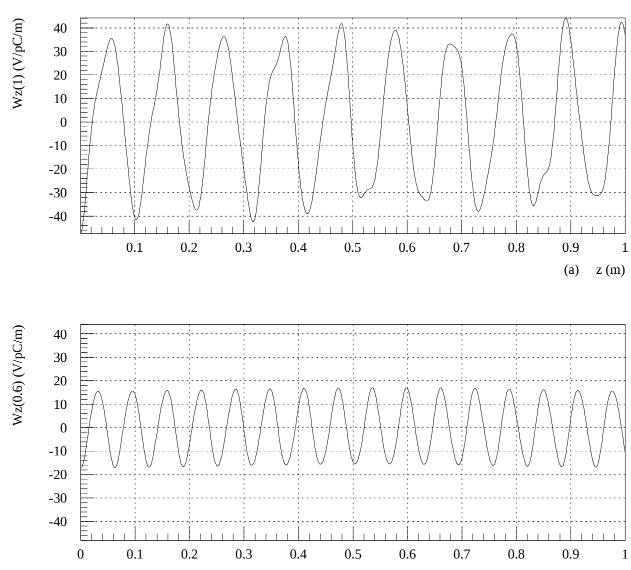


Figure 5: A periodic disk-loaded structure: $a = 0.01$ m, $h = 0.0292$ m, $D = 0.035$ m, and $\sigma_z = 0.01$ m. The long range longitudinal wakefields at (a) $\beta = 1$, and (b) $\beta = 0.6$.