# INTENSE CORKSCREWING AND WOBBLING ELLIPTIC BEAMS IN A PIECEWISE UNIFORM MAGNETIC FOCUSING FIELD

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*Abstract*

It is shown that intense corkscrewing and periodic wobbling elliptic beam equilibria exist in piecewise uniform magnetic fields. The envelope stability analysis reveals that the corkscrewing elliptic beams in a uniform magnetic field are stable, whereas the periodic wobbling elliptic beams are stable in certain regions in the parameter space. These results are useful not only in beam matching, but also in producing large-aspect-ratio sheet beams for high energy accelerators such as the next linear collider as well as for use in high-power rf sources such as sheet-beam klystrons.

## 1. INTRODUCTION

Sheet beams with large aspect ratios are required for high-energy accelerators such as the next linear collider. They are also attractive in the development of high-power rf sources because they have a smaller effective beam perveance than a round beam used in convention high-power microwave sources.

Recently, it has been shown [1] that there exists a novel class of cold-fluid corkscrewing elliptic beam equilibrium in a general focusing channel consisting of solenodal and magnetic quadrupole focusing fields. In the cold-fluid corkscrewing elliptic beam equilibrium, the transverse beam cross-section is elliptic, and it rotates as the beam propagates along the focusing channel. The internal flow velocity profile is a combination of both the elliptical-like rotating flow and quadrupole-like flow. Applications of corkscrewing elliptic beams include beam manipulation such as orienting beam ellipses at the interaction point in a high-energy collider or at a heavy ion fusion target.

In this paper, we show that intense corkscrewing and wobbling elliptic beam equilibria exist in piecewise uniform magnetic fields. Our envelope stability analysis reveals that the corkscrewing elliptic beams in a uniform magnetic field are stable, whereas the wobbling elliptic beams are stable in certain regions in the parameter space.

## 2. GENERALIZED BEAM ENVELOPE EQUATIONS

For a continuous, ultrahigh-brightness, space-charge-dominated corkscrewing elliptic beam propagating with constant axial velocity $\beta_b c\hat{\mathbf{e}}_z$ through in a focusing magnetic field

$$\mathbf{B}_0(\mathbf{x}) = B_z(s)\,\hat{\mathbf{e}}_z - \frac{1}{2}B_z'(s)\left(x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y\right), \tag{1}$$

the generalized envelope equations can be expressed [1,2]

$$\begin{aligned}
\frac{dx_1}{ds} &= x,\\
\frac{dx_2}{ds} &= -x_1\frac{d}{ds}\sqrt{\kappa_z(s)} - 2x_3\sqrt{\kappa_z(s)} - \frac{x_2x_3}{x_1},\\
\frac{dx_3}{ds} &= 2x_2\sqrt{\kappa_z(s)} + \frac{4K}{x_1} + \frac{x_2^2}{x_1},\\
\frac{dx_4}{ds} &= x_6,\\
\frac{dx_5}{ds} &= -x_4\frac{d}{ds}\sqrt{\kappa_z(s)} - 2x_6\sqrt{\kappa_z(s)} - \frac{x_5x_6}{x_4},\\
\frac{dx_6}{ds} &= 2x_5\sqrt{\kappa_z(s)} + \frac{x_5^2}{x_4},\\
\frac{d\theta}{ds} &- \frac{x_1x_5 + x_2x_4}{2x_1x_4} = 0.
\end{aligned} \tag{2}$$

In (1) and (2), $B_z'(s) = (\partial B_z/\partial s)_0$, $s = z$ is the axial coordinate, $\sqrt{\kappa_z(s)} = qB_z(s)/2\gamma_b\beta_b mc^2$ is the focusing parameter, and $K = 2q^2N_b/\gamma_b^3\beta_b^2mc^2$ is the normalized self-field perveance, where $m$ and $q$ are the rest mass and charge of the particle, $N_b$ is the number of particles per unit axial length, and $\gamma_b = \left(1-\beta_b^2\right)^{-1/2}$ is the relativistic mass factor. The variable $\theta(s)$ is the angle of the beam ellipse, and the variables $x_i$ are related to the variables $a(s)$, $b(s)$, $\alpha_x(s)$, $\beta_x(s)$ defined in [1] by

$$\begin{aligned}
x_1 &= a + b,\\
x_2 &= a\alpha_y + b\alpha_x,\\
x_3 &= \frac{dx_1}{ds},\\
x_4 &= a - b,\\
x_5 &= a\alpha_y - b\alpha_x,\\
x_6 &= \frac{dx_4}{ds},
\end{aligned} \tag{3}$$

The variables $a(s)$, $b(s)$, $\alpha_x(s)$, $\beta_x(s)$ and $\theta(s)$ fully specify the density and flow velocity of the cold-fluid corkscrewing elliptic beam equilibrium.

## 3. CORKSCREWING ELLIPTIC BEAM IN A UNIFORM MAGNETIC FIELD

For a uniform-focusing magnetic field with $\kappa_z(s) = \kappa_{z0} =$ const., equation (2) is already split into two sets of uncoupled equations, and the variable $\theta$ is a

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slaved variable. $(x_1, x_2, x_3)$ describes symmetric modes with the envelopes $a$ and $b$ oscillating in phase, $(x_4, x_5, x_6)$ describes anti-symmetric modes with the envelopes $a$ and $b$ oscillating with opposite phase.

The steady-state solutions to (2) can be obtained analytically [2]. Two branches of physically acceptable special solutions are [2]:

$$\begin{aligned} x_1 &= -\frac{4K + x_2^2}{2x_2\sqrt{\kappa_{z0}}} = const, \\ x_4 &= const, \\ x_3 &= x_5 = x_6 = 0 \end{aligned} \tag{4}$$

for branch A (i.e., A-mode), and

$$\begin{aligned} x_1 &= -\frac{4K + x_2^2}{2x_2\sqrt{\kappa_{z0}}} = const, \\ x_4 &= -\frac{x_5}{2\sqrt{\kappa_{z0}}} = const, \\ x_3 &= x_6 = 0 \end{aligned} \tag{5}$$

for branch B (i.e., B-mode). Since the equilibrium flow velocity is

$$\frac{\mathbf{V}_\perp(\mathbf{x}_\perp, s)}{\beta_b c} = \frac{x_2}{2}\left(\frac{\tilde{x}}{a}\hat{\mathbf{e}}_{\tilde{y}} - \frac{\tilde{y}}{b}\hat{\mathbf{e}}_{\tilde{x}}\right) + \frac{x_5}{2}\left(\frac{\tilde{y}}{b}\hat{\mathbf{e}}_{\tilde{x}} + \frac{\tilde{x}}{a}\hat{\mathbf{e}}_{\tilde{y}}\right), \tag{6}$$

where $(\tilde{x}, \tilde{y})$ is the coordinate in the body frame, the variables $x_2$ and $x_5$ can be considered as measures of elliptical-like rotation and quadrupole-like flow, respectively. For an A-mode, $x_5 = 0$, and the corresponding flow is pure elliptical-like rotation. A B-mode has a mixture of both elliptical-like rotation and quadrupole-like flow because both $x_2$ and $x_5$ are nonzero.

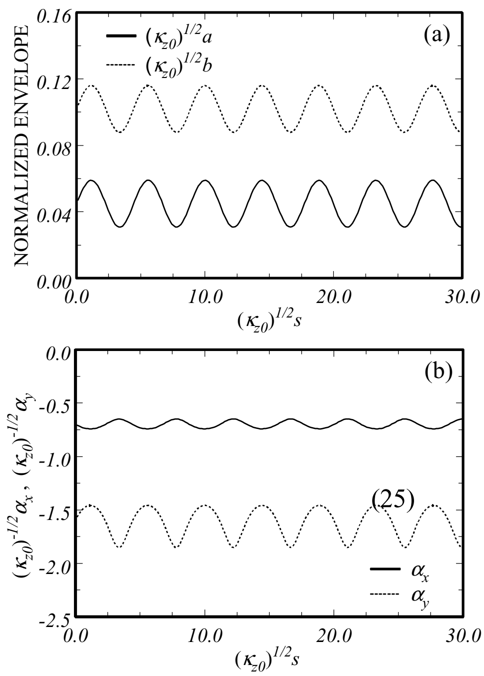
In the envelope stability analysis, we find that the eigenvalues for A-mode are [2]:

$$\begin{aligned} \lambda_{1,2} &= 0 \\ \lambda_{3,4} &= \pm 2\sqrt{\kappa_{z0}}\, i \\ \lambda_{5,6} &= \pm 2\sqrt{\kappa_{z0}}\, \frac{\sqrt{F_A}}{\hat{\alpha}_x + \hat{\alpha}_y} i \end{aligned} \tag{7}$$

and for B-mode,

$$\begin{aligned} \lambda_{1,2} &= 0 \\ \lambda_{3,4} &= \pm 2\sqrt{\kappa_{z0}}\, i \\ \lambda_{5,6} &= \pm 2\sqrt{\kappa_{z0}}\, \frac{\sqrt{F_B}}{\left|\hat{\alpha}_x + \hat{\alpha}_y + 4\right|} i \end{aligned} \tag{8}$$

where $F_A = (\hat{\alpha}_x + \hat{\alpha}_y + \hat{\alpha}_x\hat{\alpha}_y)^2 + \hat{\alpha}_x^2\hat{\alpha}_y^2$, $F_B = 2\hat{\alpha}_x^2\hat{\alpha}_y^2 + 6\hat{\alpha}_x^2\hat{\alpha}_y + 5\hat{\alpha}_x^2 + 6\hat{\alpha}_y^2\hat{\alpha}_x + 18\hat{\alpha}_x\hat{\alpha}_y + 16\hat{\alpha}_x + 5\hat{\alpha}_y^2 + 16\hat{\alpha}_y + 16$, $\hat{\alpha}_x = \alpha_x/\sqrt{\kappa_{z0}}$, and $\hat{\alpha}_y = \alpha_y/\sqrt{\kappa_{z0}}$. Since $F_A \geq 0$ and $F_B > 0$, all of the eigenvalues in (7) and (8) satisfy the condition $\mathrm{Re}(\lambda) = 0$, which means that both A- and B-mode are always stable.




(30)

Figure 1: Eigenmode oscillations for an A-mode.

Figure 1 shows the eigenmode oscillations about an equilibrium solution in an A-mode, as obtained by integrating (1) numerically. The choice of system parameters and initial conditions in Fig. 1 corresponds to: $K = 5.0\times10^{-3}$, $\alpha_x(0)/\sqrt{\kappa_{z0}} = -0.7$, $\alpha_y(0)/\sqrt{\kappa_{z0}} = -1.6$, $a(0)\sqrt{\kappa_{z0}} = 0.043$, $b(0)\sqrt{\kappa_{z0}} = 0.098$, $a'(0) = b'(0) = 0.02$. Here, the equilibrium solution corresponds to: $\alpha_x(s)/\sqrt{\kappa_{z0}} = -0.7$, $a'(s) = b'(s) = 0$, $\alpha_y(s)/\sqrt{\kappa_{z0}} = -1.6$, $a(s)\sqrt{\kappa_{z0}} = 0.043$, $b(s)\sqrt{\kappa_{z0}} = 0.098$, and. In this case, the envelopes $a$ and $b$ oscillate exactly in phase, but the variables $\alpha_x$ and $\alpha_y$ oscillate out of phase. The normalized frequency of the eigenmode oscillations is $|\lambda_5|/\sqrt{\kappa_{z0}} = 1.42$, which is in agreement with the expression for the eigenvalue $\lambda_5$ (or $\lambda_6$) given in (7).

## 4. WOBBLING SHEET BEAM IN A PERIODIC PIECEWISE UNIFORM FIELD

We make use of the generalized beam envelope equations (2) and the equilibrium solutions for a uniform-focusing magnetic field in (4) and (5) to show that it is possible to "kick" a steady-state solution from A-mode to B-mode with a magnetic reversal and vise versa to create a periodic wobbling sheet beam in a periodic piecewise uniform magnetic field. The orientation angle $\theta$ of such a wobbling sheet beam oscillates between small angles $\pm\Delta\theta$. In particular, for a periodic piecewise uniform magnetic field with

$$\sqrt{\kappa_z(s)} = \begin{cases} f_0, & (-l_0 < s < 0) \\ -f_0, & (0 < s < l_1) \end{cases} \tag{9}$$

in one period from $s = -l_0$ to $l_1$, and $\sqrt{\kappa_z(s+S)} = \sqrt{\kappa_z(s)}$, where $S = l_0 + l_1$ is the period, we find that the beam envelope is given in one period by

$$\mathbf{x}(s) = \mathbf{x}^- + [\mathbf{x}^+ - \mathbf{x}^-]\Theta(s), \tag{10}$$

$$\theta(s) = \begin{cases} (1 + 2s/l_0)\Delta\theta, & (-l_0 < s < 0) \\ (-1 + 2s/l_1)\Delta\theta, & (0 < s < l_1) \end{cases} \tag{11}$$

where $\Theta(s)$ is the unit-step function, $\mathbf{x}^-$ is defined by

$$\begin{aligned} x_1^- &= d = a + b, \\ x_2^- &= f_0 d(-1 + \sqrt{1-\zeta}), \\ x_4^- &= \eta\, d = a - b, \\ x_3^- &= x_5^- = x_6^- = 0, \end{aligned} \tag{12}$$

$\mathbf{x}^+$ is defined by

$$\begin{aligned} x_1^+ &= d, \\ x_2^+ &= f_0\, d\,(1 + \sqrt{1-\zeta}), \\ x_4^+ &= \eta d, \\ x_5^+ &= 2\,\eta f_0\, d, \\ x_3^+ &= x_6^+ = 0, \end{aligned} \tag{13}$$

and the parameters $l_0$, $l_1$, $\Delta\theta$, $\zeta$ and $\eta$ are defined by

$$\begin{aligned} l_0 &= \frac{S}{4}(1 - \sqrt{1-\zeta}), \\ l_1 &= \frac{S}{4}(3 + \sqrt{1-\zeta}), \\ \Delta\theta &= \frac{f_0 S}{8}(1 + \zeta/2 - \sqrt{1-\zeta}), \\ \zeta &= \frac{4K}{d^2 f_0^{\,2}}, \\ \eta &= \frac{a-b}{a+b}, \end{aligned} \tag{14}$$

respectively. Figure 2 illustrates the orientation angle of the periodic wobbling sheet beam equilibrium.

We have also analyzed the envelope stability of the periodic wobbling sheet beam equilibrium using the transfer matrix method, and found that it is stable in the white regions shown in Fig. 3.

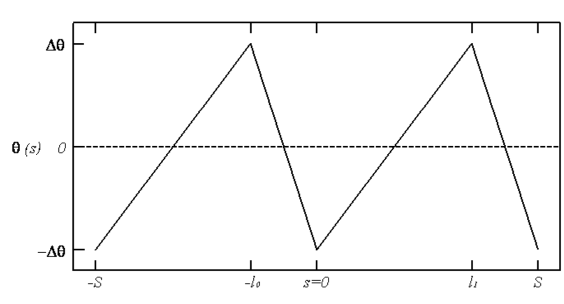



Figure 2: Axial dependence of the orientation angle of a periodic wobbling sheet beam equilibrium.

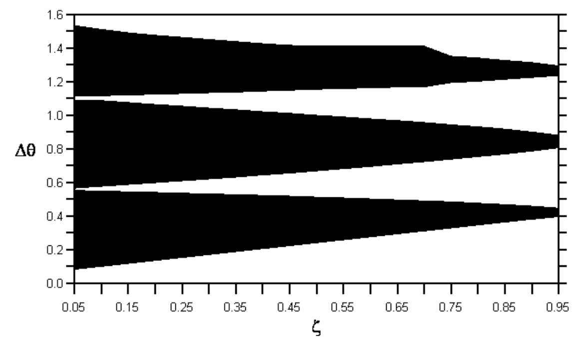



Figure 3: Plot of stable (white) and unstable (black) regions in the parameter space for a periodic wobbling sheet beam.

To illustrate the advantages of an elliptical sheet beam over a round beam in terms of required focusing field strength, we consider varying the aspect ratio $\Gamma = a/b$, while keeping the following quantities fixed: $\pi ab$, $N_b$, $\gamma_b$ and $\zeta$. Because $f_0$ is proportional to the applied magnetic field strength, we find from the definition of $\zeta$ in (14) that

$$\frac{B_\Gamma}{B_{round}} = \frac{2\sqrt{\Gamma}}{1+\Gamma}. \tag{15}$$

As an example we consider a beam with parameters similar to the SLAC 50MW, 11.4GHz, PPM klystron experiment [3], where $I_b = 190$ A, $\beta_b = 0.85$, $a = b = 2.4$ mm, RMS magnetic field $= 1.95$ kG and $S = 2.1$ cm. If we change the aspect ratio to $\Gamma = 10$, then $a = 7.8$ mm, $b = 0.76$ mm, the RMS magnetic field is reduced to 1.1 kG, and $\Delta\theta = 5.5°$.

## 5. CONCLUSION

We showed that intense corkscrewing and wobbling elliptic beams equilibria exist in piecewise uniform magnetic fields. The envelope stability analysis revealed that the corkscrewing elliptic beams in a uniform magnetic field are stable, whereas the wobbling elliptic beams are stable in certain regions in the parameter space. These results are useful not only in beam matching, but also in producing large-aspect-ratio sheet beams for high energy accelerators such as the next linear collider as well as for use in high-power rf sources such as sheet-beam klystrons.

This work was supported by the DOE, HEP, Grant No. DE-FG02-95ER-40919, and in part by the DOE, OFES.

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