WAVE PROPAGATION IN A FRACTAL WAVE GUIDE

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Abstract

We analyze the propagation of electromagnetic waves in a wave guide that has the shape of Koch's snowflake, a well-known fractal.

INTRODUCTION

The investigation of Radio Frequency wave propagation in wave guides with rough surfaces has been carried out since a long time by considering random boundary conditions and solving the Maxwell's equations in the statistical sense of the boundaries [1–4]. The effect of a naturally rough surface would be that the perimeter is infinite [5]. In this paper we emulate the roughness of the perimeter by means of a fractal, the Koch's snowflake, which has increasing perimeter through the generations but a finite cross-sectional area. This allows us to put a controlled rough surface on the waveguide with each progressive generation of the fractal and systematically investigate the effect of "roughness". Since the snowflake has many corners into which modes can enter, we expect that there are more higher modes compared to to a smoother waveguide such as a circular one. Therefore we try to address the question of how much faster the number of higher modes grows as a function of the order of the snowflake. Since the fractal snowflake is such a peculiar shape with asymptotically finite area, but infinitely growing perimeter as a function of order, we explore different ways to characterize this growth.

We start with the source free Maxwell's equations in vacuum $\vec{\nabla} \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$ and $\vec{\nabla} \times \vec{H} = \epsilon \frac{\partial \vec{E}}{\partial t}$ and derive the wave equations for the cylindrical co-ordinate system. Then assuming the harmonic propagation of the wave in z-direction with frequency ω and propagation constant β and looking for the TM modes we arrive at the second-order partial differential equation (PDE)

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) E_z = \left(\beta^2 - k^2\right) E_z \qquad (1)$$

where $k^2 = \omega^2/c^2$ is the wave number, c is the speed of light and E_z is the electric field in the longitudinal direction. Using the cylindrical co-ordinate system for the circular wave guide and the separation of variables the eq. (1) can be separated into two decoupled second order differential equations which can be further represented in matrix form as a system of 4 first order differential equations. The eigenvalues of this system, coupled with the boundary condition $E_z = 0$ at radius, r, of the circular waveguide, will produce the dispersion relation for the circular waveguide as shown in eq. (2)

$$\frac{\omega^2}{c^2} - \beta^2 = \frac{p_{nm}^2}{r^2} \qquad (2)$$

where p_{nm} is the m-th root of the n-th Bessel function $J_n(x)$. The number of eigenmodes for a circular waveguide is thus the total number of all p_{nm} that stays below a certain value. The circular waveguide forms a good starting point as it has no corners. The modes thus form a base line as the problem of modes that fill up corners does not arise in this case. The snowflake, however, has lots of corners.

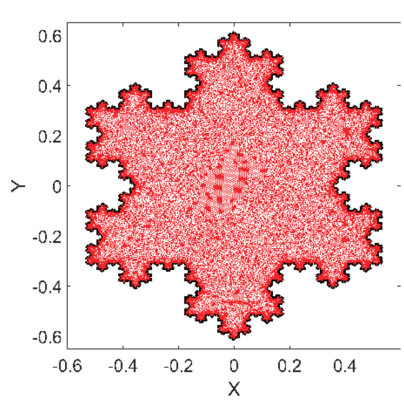

Figure 1: Cross-section of a snowflake waveguide with inlaid mesh for PDE solution.

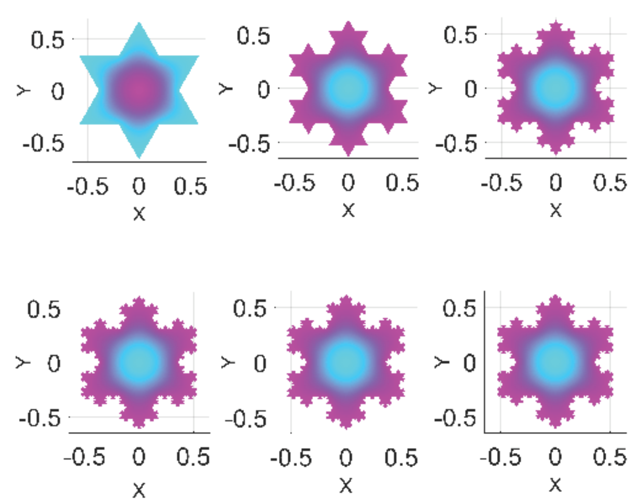

Figure 2: The first resonant mode of a snowflake waveguide with increasing generation.

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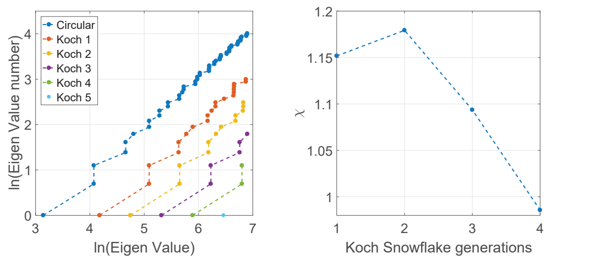

Figure 3: The eigenvalues and the fit parameter with snowflake generations with constant perimeter.

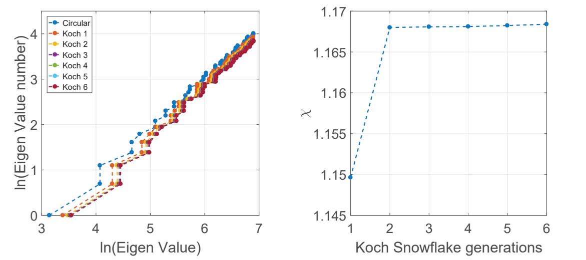

Figure 5: The eigenvalues and fit parameter for Koch's snowflake with cross-section of constant area.

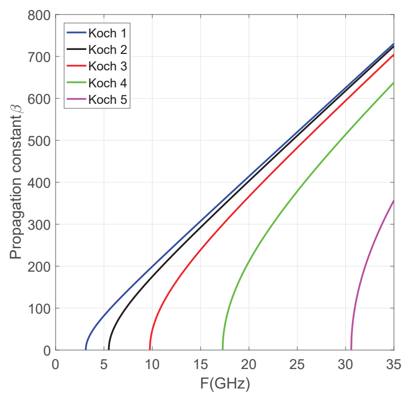

Figure 4: The dispersion relation of waveguide with increasing generation of the Koch's snowflake as cross-section with constant perimeter.

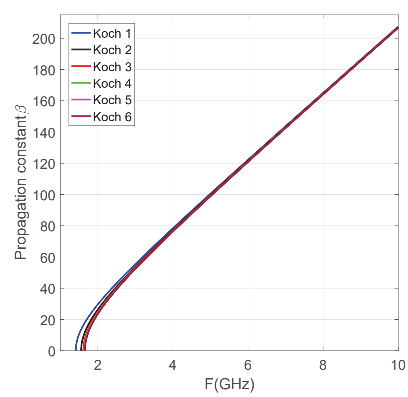

Figure 6: The dispersion relation of waveguide with increasing generation of the Koch's snowflake as cross-section of constant area.

KOCH'S SNOWFLAKE WAVEGUIDE

Changing from the circular wave guide to the one with the Koch's snowflake as the cross-section keeps eq. (1) and the rest of the procedure the same, but changes the boundary conditions. The cross-section and the inlaid mesh to solve the eigenvalue problem are shown in Fig. 1. The first resonant mode for the same is shown in Fig. 2. We first investigate the snowflake first by keeping the perimeter constant and next keeping the area constant through the snowflake generations.

Constant Perimeter

The perimeter is kept equal to that of the circular waveguide used for reference and is kept constant by adjusting the length of the sides of the starting triangle. The eigenvalues of the solution of the corresponding PDE problem is shown on the left in Fig. 3 while the dispersion relation obtained is shown in Fig. 4. Since the growth of the eigenvalues in Fig. 3 in double-logarithmic presentation is rather linear we use the following parameterization for the curves

$$N\left(<\frac{f}{f_0}\right) \propto \left(\frac{f}{f_0}\right)^{2\chi} \tag{3}$$

where N is the cumulative density distribution function of the number of eigenmodes with a cutoff frequency smaller than f. Here we normalize by the fundamental mode frequency f_0. The values of the fit parameter χ are shown on the right of Fig. 3.

Constant Area

The cross-sectional area is kept equal to that of the circular waveguide used for reference and is kept constant by adjusting the length of the sides of the starting triangle. The eigenvalues of the solution of the corresponding problem is shown on the left of Fig. 5 while the dispersion relation obtained is shown in Fig. 6. We do a similar fit as eq. (3) for this case as well and the fitted parameters are plotted on the right of Fig. 5.

Comparing Fig. 3 and Fig. 5 we can see that the change of eigenvalues is larger for the case of the constant perimeter compared to that of the constant area. Also from Fig. 4 and 6 we see the difference in behavior of the fit parameter for the two cases. While the parameter decreases for the case of the constant perimeter, it is seen to saturate to a value of 1.168 for the case of the constant area.

Cutoff with First Mode Constant

Finally we investigate how the number eigenvalues and thus the modes below a given cutoff frequency develops when keeping the frequency of the fundamental mode constant. Basically we always divide the eigenvalues by the first one. The resulting plot is shown on the top of Fig. 7. Again we see an apparent linear relationship that allows us to deter-

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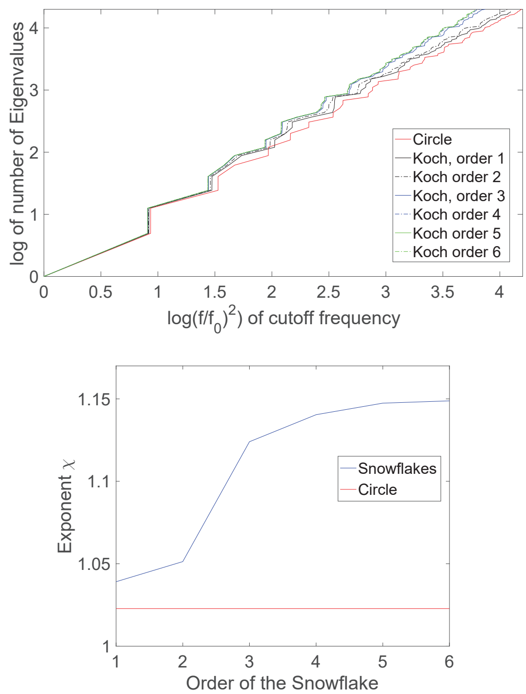

Figure 7: The number of eigenvalues below a given cutoff versus the cutoff while keeping the fundamental mode cutoff frequency fixed is shown on the top and the fit parameter χ on the bottom.

mine the exponent χ which we show on the bottom in Fig. 7. For comparison we also added the eigenvalue spectrum for the circular waveguide as a red line in both plots.

We observe that the slope of the curves for the Koch snowflake is always bigger than that of the circular waveguide indicating that there are more modes that can 'creep into' the many corners of the snowflake, especially at higher frequencies. The snowflake becomes overmoded more quickly compared to a circular waveguide and the exponent χ serves as a parameter to quantify the 'speed', by which it becomes overmoded.

In this context it is instructive to compare the arbitrarily chosen eigenmode number 70 for a a Koch snowflake of order 6 and that of a circular waveguide. We show the corresponding modes in Fig. 8 where we observe that the mode for the snowflake really occupies the corners of the snowflake.

CONCLUSION

By numerically investigating the cumulative density distribution function $N(< f / f_0)$ of the snowflake and a circular waveguide for comparison we found a useful measure to explore and quantify the speed by which the waveguide becomes overmoded.

It should be noted that during the first generations of the fractal the change of shape is large enough to be considered a dent, rather than roughness. While a random roughness will keep the cross-sectional area constant, a dent will keep

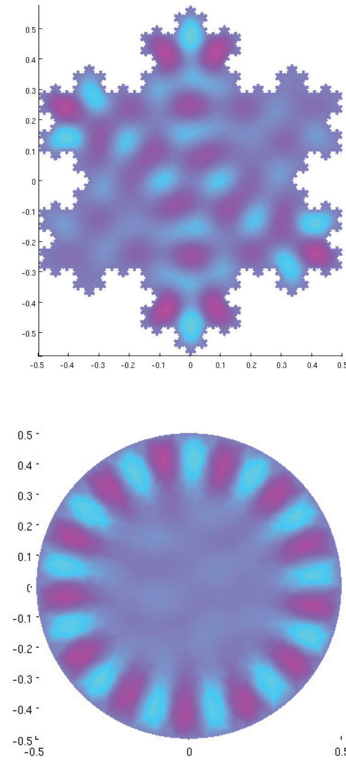

Figure 8: Comparing mode number 70 for a Koch snowflake of order 6 (top) and of a circular waveguide (bottom). The modes 'creeping into' the corners are clearly visible.

the perimeter constant while the area will change. From the investigation we can see that the change in dispersion relation is larger for the constant perimeter case (dent) than the constant area case. This points to the idea, that these two cases in conjunction may be useful to investigate and model the roughness of waveguides.

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