NOVEL IMPLEMENTATION OF NON-PARAMETRIC DENSITY ESTIMATION IN MICE*

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Abstract

Cooled muon beams are essential for a future Neutrino Factory or Muon Collider. The international Muon Ionization Cooling Experiment (MICE) aims to demonstrate muon beam cooling through ionization energy loss of muons in material. Figures of merit for muon cooling are the transverse root-mean-square (RMS) emittance or phase-space volume reduction and phase-space density increase. In this paper, kernel density estimation (KDE) and nearest neighbor density estimation (NNDE) techniques are introduced and the implementation of the KDE method in MICE, as a measure of phase-space density and volume, is described.

MUON IONIZATION COOLING

Ionization cooling occurs when muons lose momentum as a result of interactions with the atomic electrons of an absorbing material. In addition to ionization energy loss, multiple collisions with the atomic nuclei of the material can occur. This process is referred to as multiple Coulomb scattering [1] and it causes small-angle deviations in the trajectories of muons. For effective ionization cooling, the ionization energy loss should dominate the multiple Coulomb scattering process in an absorber. The two competing processes cause a change in the normalized transverse RMS emittance [2],

$$\frac{d\varepsilon_\perp}{ds} \simeq -\frac{\varepsilon_\perp}{\beta^2 E_\mu}\left\langle\frac{dE}{ds}\right\rangle + \frac{\beta_\perp(13.6\mathrm{MeV}/c)^2}{2\beta^3 E_\mu m_\mu X_0}, \qquad (1)$$

where βc, E_μ, and m_μ are the muon velocity, energy, and mass, $\langle \frac{dE}{ds}\rangle$ the magnitude of the mean ionization energy loss rate, X_0 the absorber radiation length, $\beta_\perp$ the transverse beta function at the absorber, and $\varepsilon_\perp$ the input transverse emittance. MICE studies beam cooling for various considered values of input emittance, momentum, and beta function. An example running configuration, studied in this paper, is 6 π mm·rad input emittance, 140 MeV/c momentum, and ~ 800 mm beta function at the center of the absorber.

To measure the change in normalized emittance as well as the phase-space volume or the phase-space density, MICE uses two tracking detectors, immersed in constant fields of the Spectrometer Solenoid modules (Fig. 1). Each tracker is composed of five scintillating fiber stations. Muons traversing the trackers form helical tracks and their individual positions and momenta are reconstructed via pattern recognition and Kalman fitting algorithms [3].

NON-PARAMETRIC DENSITY ESTIMATION

Non-parametric density estimation (DE) techniques estimate the phase-space density without any assumptions about the functional form of the particle distribution. This allows individual data points to "speak for themselves". In MICE, single particle measurements are made of transverse position and kinetic momentum coordinates of each muon, (X, P_x, Y, P_y). These coordinates are coupled in the presence of solenoidal fields and are the inputs to the phase-space density estimation.

The kernel density estimation (KDE) technique estimates the phase-space density at any arbitrary point, $\vec{r} = (x, p_x, y, p_y)$ (also known as a reference point) by summing over the contributing kernels (weight functions of certain widths) centered at each i^{th} muon in the transverse position-momentum phase space $\vec{R}_i = (X_i, P_{xi}, Y_i, P_{yi})$ [4],

$$\hat{f}(\vec{r}) = \frac{1}{nh^q\sqrt{(2\pi)^q}}\sum_{i=1}^{n} K\left[\frac{\vec{r}-\vec{R}_i}{h}\right]. \qquad (2)$$

In Eq. 2, q is the dimension, h is the parameter which tunes the level of smoothing of the density curve (known as bandwidth parameter), and n is the number of muons in the sample. The kernel functions used in this analysis are of Gaussian form [4],

$$K\left(\frac{\vec{r}-\vec{R}_i}{h}\right) = \frac{1}{(2\pi)^{\frac{q}{2}}}\exp\left(-\frac{\left(\vec{r}-\vec{R}_i\right)^2}{2h^2}\right). \qquad (3)$$

The nearest-neighbor density estimation, NNDE (known as the k^{th} nearest neighbor in the literature [4]) can be written similarly to Eq. 2:

$$\hat{f}(\vec{r}) = \frac{1}{nd^q\sqrt{(2\pi)^q}}\sum_{i=1}^{n} K\left[\frac{\vec{r}-\vec{R}_i}{d}\right], \qquad (4)$$

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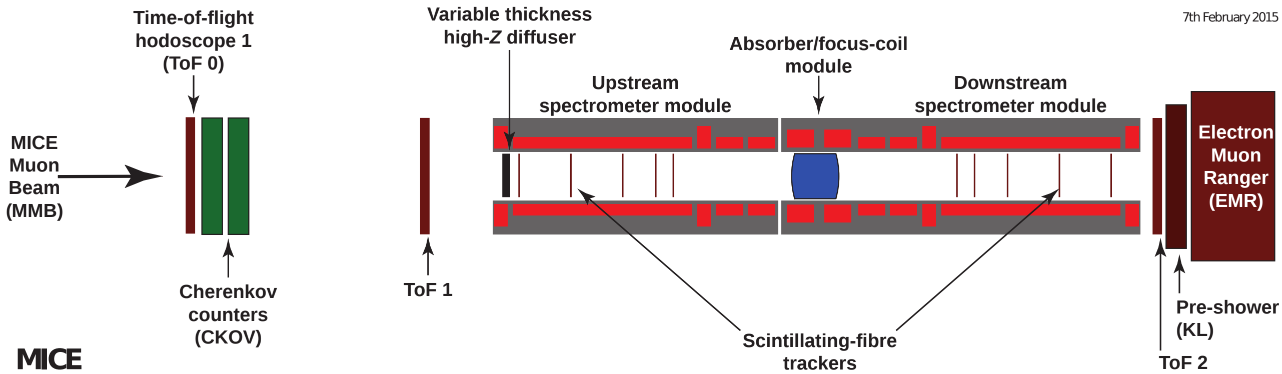

Figure 1: Schematic diagram of the Muon Ionization Cooling Experiment in its current Step IV configuration.

where d represents the distance between $\vec{r}_i$ and its k^{th} nearest neighbor. In NNDE, the level of smoothing of the density curve varies based on particle position in the distribution: distances are larger for the muons near the periphery of the multi-dimensional distribution compared with those near the core of the beam.

The aim of DE is to minimize the discrepancy between the true (predicted) density and the estimated density. The commonly used measure of such discrepancy is the mean integrated square error (MISE) [4],

$$\text{MISE} = \left\langle \int \left\{ \hat{f}(\vec{r}) - f(\vec{r}) \right\}^2 d\vec{r} \right\rangle, \tag{5}$$

where $\hat{f}(\vec{r})$ and $f(\vec{r})$ represent the estimated density and the true density.

MISE can be simplified to a sum of the squared bias (error associated with the level of smoothing) and variance (error associated with the level of noise) of the estimated density [4]. DE techniques often suffer from a bias−variance trade off: if an attempt is made to reduce the bias ($\propto h^2$) by selecting a small bandwidth parameter, h, the variance ($\propto \frac{1}{h}$) increases. This is shown in Fig. 2, where the true probability density function or PDF is a Gaussian distribution with $10,000$ data points. To illustrate the trade-off feature, density curves with different bandwidth parameter values are compared. One way to obtain an optimal bandwidth parameter is to minimize Eq. 5. Such optimal h depends on the true density and, if chosen to be Gaussian, is referred to as the normal reference bandwidth. It depends on the sample size, the number of dimensions, q, and Σ, the covariance matrix representing the correlations between the coordinates: $h = \left(\frac{4}{q+2}\right)^{\frac{1}{q+4}} n^{-\frac{1}{q+4}} \Sigma$. In Fig. 2 (upper plot), the normal reference h is 0.005, which reveals a density curve close to the true Gaussian PDF. If this optimal h is increased by a factor of 10, the density curve is overly smooth, and if decreased by the same factor, the density curve gets noisier. The NNDE curves (lower plot in Fig. 2) also show a bias−variance trade off. The parameter that determines the level of smoothing in NNDE is k and the optimal value is 100 for a distribution of size $10,000$ [4]. This optimal value results in a noisy (bias-reducing) curve that closely follows the true PDF curve. However, if the farthest neighbor is selected, the density curve becomes overly smooth.

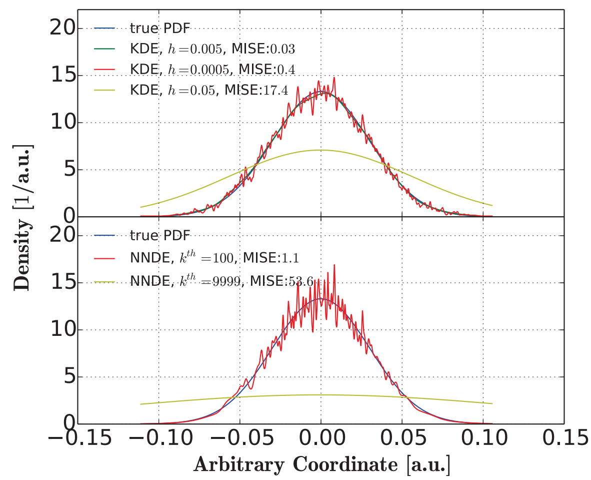

Figure 2: Plot of estimated density vs. an arbitrary coordinate of a Gaussian sample.

SIMULATION RESULTS

To simulate a MICE data sample, an input beam is generated using Monte Carlo simulation routines in MICE Analysis User Software (MAUS) and Xboa [5]. The initial beam is Gaussian and is matched to the 3 T field of the upstream Spectrometer Solenoid. The input beam emittance and momentum are chosen as 6.0 π mm·rad and 140 MeV/c. This is input into G4beamline [6] for tracking of $100,000$ muons from the upstream to the downstream tracker. The currents in the Spectrometer Solenoid modules are based on some recent MICE data runs. The transmission efficiency of this simulated lattice is about 85%. Some muons are lost due to scraping and optical mismatch after passing through the absorber. Unlike [8, 9], no transmission selection is applied to discard them from the sample. The phase-space density is predicted to increase at the beam core [7–9]. As a result, the density is estimated for the core muons only (muons within the 1σ of the beam).

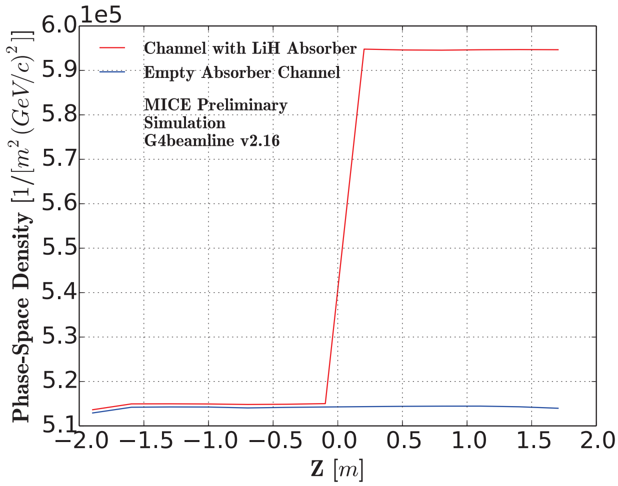

Figure 3: Evolution of the phase-space density in a simulated MICE lattice.

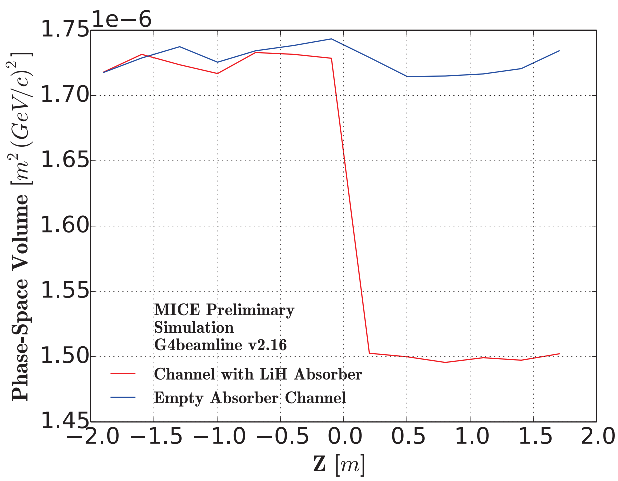

Figure 4: Evolution of the phase-space volume in a simulated MICE lattice.

Figures 3 and 4 display the evolution of the phase-space density and volume approximately corresponding to the 9^{th} percentile contour across the absorber and across a channel without an absorber. The 9^{th} percentile contour in four dimensions approximately represents 1σ of the beam and thus contains 9% of the sample. The phase-space coordinates of each muon are obtained from G4beamline NTuple hits.

The 9^{th} percentile contour is tracked from the upstream tracker reference plane (the station closest to the absorber), passing through a 65 mm LiH absorber (centered at $z = 0$ m), to the downstream tracker reference plane. An increase in phase-space density (average phase-space density within the 9^{th} percentile contour) and reduction in phase-space volume at the downstream plane indicate cooling. In the case of a channel without an absorber, no changes in density and volume are observed. To compute the phase-space volume, a separate Monte Carlo (MC) technique is used [7–9]. The slight fluctuations in the volume curves are statistical. The density and volume are estimated using the normal reference bandwidth parameter.

CONCLUSION AND FUTURE PLAN

The KDE technique has been used to characterize muon beam cooling for the simulated MICE Step IV lattice. It gives a detailed diagnostics of the muon beam as it traverses a material: the density increases within the core due to beam cooling [7–9]. Application of KDE to MICE data along with a detailed systematics analysis is in progress. To overcome the statistical noise in nearest neighbor density estimation, development of a nearest neighbor-KDE hybrid technique (adaptive KDE) is underway along with a new bandwidth selection approach (log-likelihood cross validation) which unlike normal reference rule is not obtained based on an assumption about the underlying distribution [4].

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