Impedance Calculation above Cut-Off with URMEL-I

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ABSTRACT

URMEL-I is a computer code to calculate the impedance of obstacles of arbitrary but cylindrically symmetric shape with side tubes - especially thought for the impedance calculation above cut-off where existing resonator codes cannot be used. The numeric solution is based on the FIT-method. It affords the solution of an inhomogeneous, complex matrix equation for a given frequency. A multigrid algorithm has been set up to solve the linear system. This special solver is still in the stage of development but already giving results which are shown here.

INTRODUCTION

Some important aspects in the design of accelerating components are their influence on the beam and their own behaviour in the working machine. The calculation of the impedance, which describes the wake force in the frequency domain, and the evaluation of the wake potential are equivalent ways of studying these questions. In this paper a numerical method will be presented to calculate the longitudinal impedance for a cylindrically symmetric structure, as e.g. a cavity with beam ports.

Cavities as components of an accelerator have only a finite number of resonant modes below the lowest cut-off frequency for travelling waves in the beam port. Above this cut-off frequency no ideal resonance can exist since fields may travel out of the cavity. Nevertheless so-called quasi resonances may build up. Above cut-off the non-zero values of the impedance form a continuous spectrum. A peak in this spectrum can be considered a resonance.

There are a number of computer codes to evaluate the lowest resonant frequencies and quality factors for cavities needed for the impedance calculation below cut-off, e.g. [1], [2]. Recently Gluckstern and Neri adapted the resonator code SUPERFISH [3] for impedance calculations above cut-off [4]. The program URMEL-I calculates the impedance as a function of frequency (including the region above cut-off). For this calculation the cavity is excited with a beam current equal to the Fourier transform of a point charge traversing the structure. Solving Maxwell's equations for the fields yields the electric field and thus by integration the impedance.

DEFINITION OF THE IMPEDANCE

The current density of a point charge on the axis of a cylindrically symmetric structure equals:

$$\vec{j}(r,\varphi,z,t) = \rho(r,\varphi,z,t)\cdot\vec{v} = Qv\delta(r)\delta(z-vt)\vec{e}_z, \tag{1}$$

with $Q=$ charge and $v=$ speed of the point charge.
The longitudinal impedance of a structure is then given by:

$$Z(\omega) = \frac{1}{Q}\int_{-\infty}^{\infty} \tilde{E}_z(r=0,\varphi,z,\omega)\, e^{i(\omega/v)z}\, dz, \tag{2}$$

where $\tilde{E}_z$ is the Fourier transform of the longitudinal electric field E_z. ([5] gives more details.) So far only relativistic particles are considered, i.e. $v = c$.

In [6] it is shown that the integral can be taken at any radius r for a cylindrically symmetric structure. Thus the integral can also be taken at the tube radius where the integrand vanishes in the tube region. Consequently the evaluation of the integral only over the gap of the cavity yields the impedance.

NUMERICAL FORMULATION OF THE PROBLEM

To calculate the impedance Maxwell's equations have to be discretized. Here finite differences, in particular the FIT-method [7], are used for the discretization. An extension of this method was adopted in connection with the solution of the discretized equations. Since only the cavity region and neighboring parts of the (infinitely long) beam ports is entered as geometry an "open" boundary condition was introduced.

Maxwell's Equations for the problem

Maxwell's equations for the problem read as :

$$\oint_{\partial A} \vec{H}'\cdot d\vec{s} = i\,\frac{\omega}{c}\cdot\int_A \epsilon_r \vec{E}'\cdot d\vec{A} + I', \tag{3}$$

$$\oint_{\partial A} \vec{E}'\cdot d\vec{s} = -i\,\frac{\omega}{c}\cdot\int_A \mu_r \vec{H}'\cdot d\vec{A}, \tag{4}$$

where ∂A stands for the boundary of area A and $\vec{E}'$ and $\vec{H}'$ are normalized complex phasors (compare [5]). At present only monopole fields are treated ($\partial/\partial\varphi = 0$).

Since the structure is excited by a beam current with frequency ω the fields are composed of an inhomogeneous part that is caused by the current and the homogeneous part. Therefore H' can be written as $H' = H_\varphi^* + H_\varphi^0$ with

$$H_\varphi^*(\omega,r,z) = \frac{c\,Q}{2\pi r}e^{-ik(z-z_0)}, \tag{5}$$

with the wavenumber $k=\omega/c$ and the phase kz_0.

The open boundary condition

For a frequency above the lowest cut-off frequency of the beam port the excited wake fields can propagate in the tube with propagation constant k' (compare figure 1).

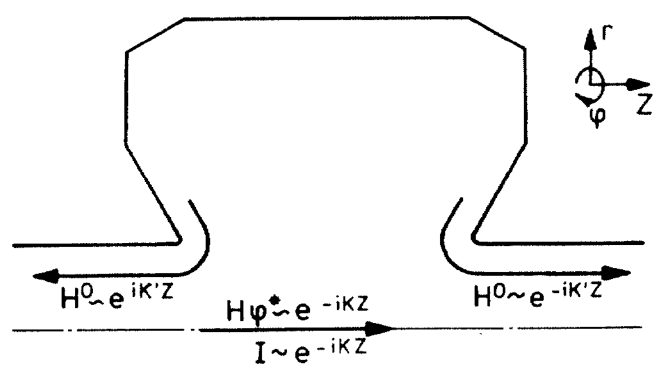

Figure 1: Longitudinal dependance of exciting current and magnetic field components.

For the numerical calculation the beam ports have to be cut at some convenient distance from the cavity. At these boundaries the reality of an open beam port has to be simulated.
From (5) it follows for H_φ^*

$$H_\varphi^*(\omega,r,z-\Delta) = H_\varphi^*(\omega,r,z)\,e^{ik\Delta} \doteq (1+ik\Delta)\,H_\varphi^*(\omega,r,z) \tag{6}$$

For the homogeneous azimuthal magnetic field the relation

$$H_\varphi^0(\omega,r,z-\Delta) \doteq (1-ik'\Delta)\,H_\varphi^0(\omega,r,z) \quad \textit{at the left} \tag{7}$$

$$H_\varphi^0(\omega,r,z+\Delta) \doteq (1-ik'\Delta)\,H_\varphi^0(\omega,r,z) \quad \textit{at the right} \tag{8}$$

is valid. k' is exactly evaluated in the frequency range just above cut off where only one type of wave can propagate, otherwise $k'=k$ i taken as approximation.
These first order equations are used to set up the difference equation for grid points at the right and left boundary (compare figure 2).

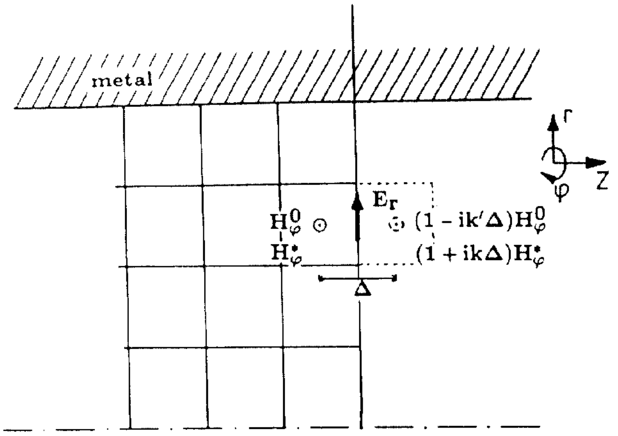

Figure 2: Use of the open boundary condition at the grid boundary in the tube to the right side of the cavity.

Discretization

Maxwell's equations are discretized with the FIT-method for a rectangular grid. This implies many analogies with URMEL [1].

The difference equation for H_φ^0 is deduced from (4) while the discretization of equation (3) gives the ones for E_r and E_z (compare [1]). For the equations for E_r at the left and right boundary of the grid the open boundary simulation is used (compare (6), (7) and (8)). The difference equations for E_z use for $r > 0$ the relation $rH_\varphi^*(\omega, r, z) = (r - \Delta)H_\varphi^*(\omega, r - \Delta, z)$ which holds because of (5); on the axis ($r = 0$) the current does not vanish but can be expressed in terms of H_φ^*.

In the difference equation for H_φ^0 the other components are replaced by their difference equations. This yields a system of linear equations with the homogeneous azimuthal field components as unknowns and the inhomogeneous ones on the right hand sides.

Extension of the FIT-method

In context with the solution method for the linear equations system an extension of the FIT-method was introduced:
A grid cell may now be partially filled with metal in any way as e.g. in figure 3. The FIT-method uses the area of a grid cell and the lengths of the sides. When a cell is partially filled with metal only the area of the remaining vacuum and the lengths of the sides not bordering metal are used.

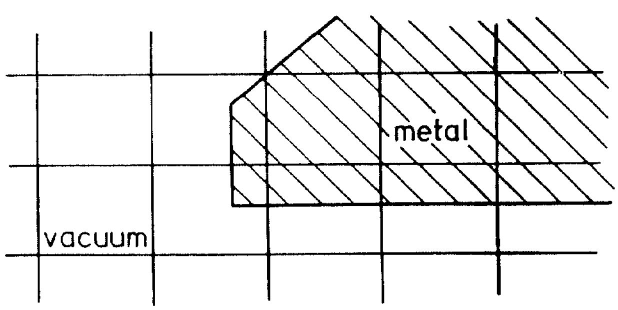

Figure 3: Partially filled grid cells.

SOLUTION OF THE NUMERICAL PROBLEM

Discretization of Maxwell's equations leads to a complex linear system of equations

$$\mathbf{L}\,\mathbf{h} = \mathbf{b} \qquad (9)$$

with $\mathbf{L} = \mathbf{A} + ik'\mathbf{D} - k^2\mathbf{I}$ and $\mathbf{b} = k^2\mathbf{h}^*$.
L is a (N x N)-matrix. The solution vector **h** holds the homogeneous azimuthal magnetic field components $(H_{\varphi 1}^0, ..., H_{\varphi N}^0)$. The matrices **A**, **D** and **I** (unit matrix) are purely real. **A** equals the matrix of URMEL [1] for the monopole case. **D** is a diagonal matrix expressing the open boundary condition. On the right hand side $\mathbf{h}^*$ equals $(H_{\varphi 1}^*, ..., H_{\varphi N}^*)$.

With **A** the matrix **L** has a band structure with only four off-diagonals. **A** can be made symmetric as is explained in [1] but **L** is non hermitian and not even positive definite. Near (quasi-)resonances **L** even becomes nearly singular. At resonances below cut-off **L** is exactly singular. Because of round-off errors this is true also near these resonances.

Multigrid algorithm as solver for the linear system

The linear system in question has a large, sparse, indefinite and non-hermitian matrix. Therefore it should be treated by a fast iterative solution method. A multigrid algorithm has been developed and is presented here.

Multigrid methods [8] have shown a big success in the fast solution of partial differential equations. For some problems they are several orders of magnitude faster than other known methods. The main idea of the multigrid methods lies in the combination of an iteration method I, which smoothes the high frequency parts of the error in a few steps, with another iteration II, that reduces the low frequency parts in the error. I is just a classical iteration method like the Gauss-Seidel method. One iteration step of II consists of a correction evaluated on a coarser grid. Even though II is not convergent the combination of I and II converges very fast.

In URMEL-I a so called Full-Multi-Grid-Method (FMG) with V-cycles is used. A V-cycle is a special kind of a single iteration step of II. Figure 5 shows schematically on two grids how this works. In practice at least three grids are usually taken (compare figure 4).

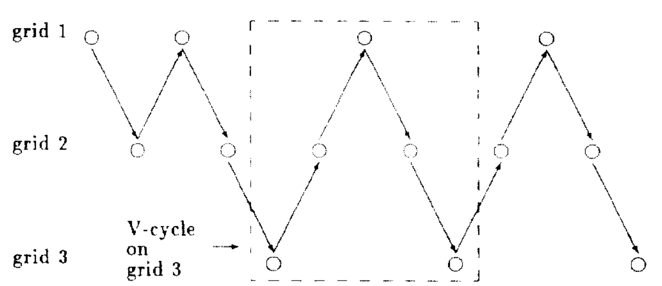

Figure 4: FMG-method with three grids and two V-cycles on each grid. Grid 1 is the coarsest grid.

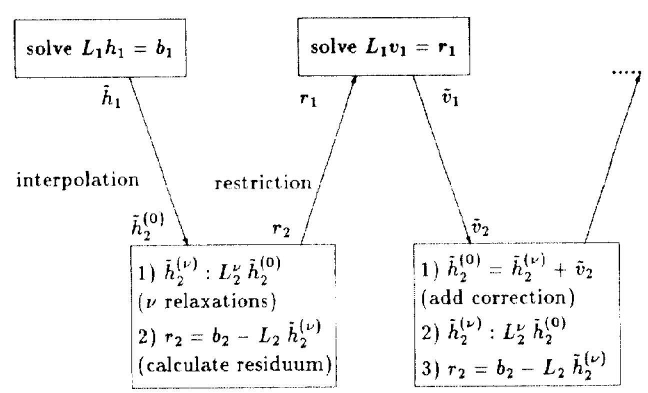

Figure 5: FMG-method on two grids.

The *coarser grids* are given by every second grid line of the grid on the next higher level, i.e. $h_{coarse} = 2h_{fine}$ for regular grids.
On each grid level the *matrix* L_i has to be set. In this process it is very important to solve the same physical problem, i.e. to treat the same geometry. Therefore the extension of the usual set up of FIT-equations on a rectangular grid, which is described above, was introduced.

The *solution on the coarsest grid* is done by LU-factorization with a LINPACK-routine [9].
As *grid transfer* either bilinear interpolation to a finer grid or corresponding restriction to a coarser grid is chosen.
As *relaxation method* the Gauss-Seidel method is taken as long as a V-cycle reduces the $\|.\|_2$-Norm of the residuum. From then on the Kaczmarz method [10] is used.
The *number of V-cycles* has not been fixed a priori but is set by a convergence criterium. Also the *number of relaxation sweeps* depends on a convergence criterium.

At the actual status of URMEL-I details of the algorithm have to be improved to get a better performance. Problems with this are caused e.g. by the near-singularity close to quasi-resonances, by the interpolation error at corners in the boundary and by the frequency dependance of the problem, which affords finer grids with increasing frequency.

In conclusion it can be said that the results are not yet satisfying, especially in their accuracy, but already show a reasonable agreement with analytical results and results obtained from wakepotential calculations with TBCI [6].

EXAMPLE

As example a pillbox with 65 mm gap, 100 mm radius and 50 mm tube radius is chosen here, compare figure 6.

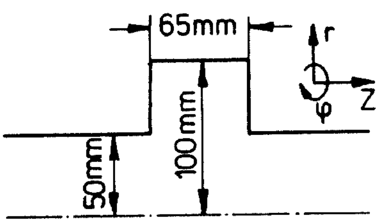

Figure 6: Pillbox cavity taken as example.

In the frequency range where a big stepsize is possible, i.e. N is relatively small, the results from LU-factorization could be compared with results from the MG-algorithm. Figure 7 shows the impedance calculated with the LU-factorization as solver for Δ_z = 4.06 mm, 8.12 mm and 16.25 mm giving 640, 160 respectively 40 vacuum cells. It can be seen that the discretization induces a compression of the impedance curve in the frequency direction. Figure 8 shows the curve of a MG-solution on two grids in comparision with the coarse grid solution and the LU-solution on the fine grid. Besides the areas with convergence difficulties as mentioned above the MG- and LU-solution are identical proving the quality of the MG-algorithm in general.

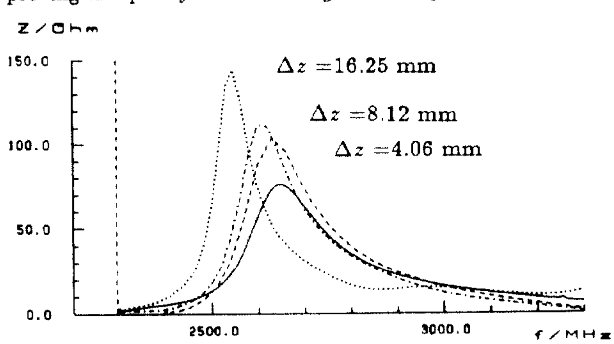

Figure 7: Real part of the impedance above cut-off (= 2295 MHz) calculated by URMEL-I with direct solution method and stepsizes Δz =4.06, 8.12, 16.25 mm and impedance by analytical solution with Henke's code.

For pillboxes with side tubes the impedance can be analytically calculated by Fourier series. H.Henke [11] used this to compute numerically the impedance for pillboxes. Henke's method was taken for comparisons showing a reasonable agreement as can be seen in figure 9.

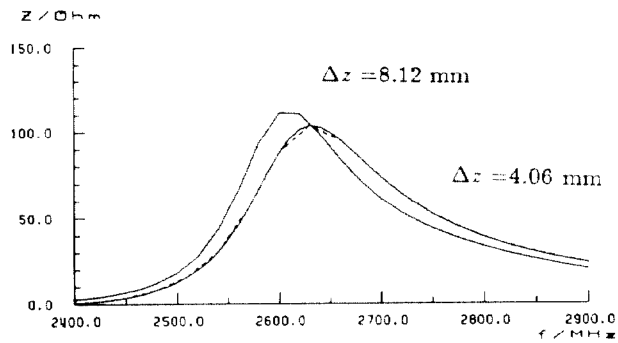

Figure 8: Comparision of real part of the impedance by direct solution on grid with Δz = 8.12 mm and by 2-grid-MG-solution respectively direct solution on grid with Δ_z = 4.06 mm (cut-off = 2295 MHz).

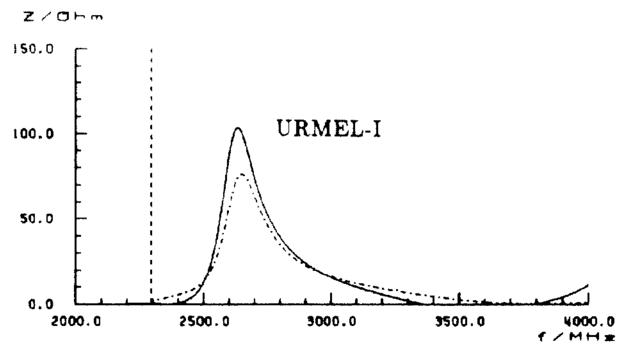

Figure 9: Real part of the impedance above cut-off (= 2295 MHz) computed by URMEL-I and by Henke's code.

SUMMARY

The code URMEL-I presents a new tool to calculate the impedance of obstacles of arbitrary but cylindrically symmetric shape with side tubes. The preliminary version gives gives reasonable results, indicating the validity of the method.

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