# THE BEAM-BEAM LIMIT AND THE DEGREE OF FREEDOM 

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#### Abstract

Luminosity in $e^{+} e^{-}$collider is limited by emittance growth due to the beam-beam interaction. Emittance is defined as phase space volume of each degree of freedom in which particles in beam occupy. The emittance growth is caused by diffusion due to the strong nonlinear force of the beam-beam interaction. The diffusion is strongly related to the number of the degree of freedom of considering dynamical system. A very high beam-beam parameter can be realized to reduce the number of degree of freedom effectively by optimizing the operating tune of an accelerator.


## INTRODUCTION

Emittance growth, which limits accelerator performance, is the most important issue of accelerator physics. Emittance is defined as an average of a half of CourantSnyder invariant $(\boldsymbol{J})$ over all beam particles, where the invariant is given for each particle in each degree of freedom in linearized system. Emittance is constant of motion in linear system from its definition; average of the invariant.

There are several possibilities, which cause the emittance growth in storage rings. Linear coupling and coherent instability are rather trivial sources. We discuss an incoherent emittance growth due to nonlinear diffusion for the beam-beam interactions in this paper. The number of degree of freedom, which is the number of canonical variable pair in Hamiltonian, plays a important role for the emittance growth [1]. We discuss the importance of the degree of freedom and a concept, "reduction of the degree of freedom", to achieve a very high beam-beam parameter in circular collider.

Another is an external diffusion which is induced by synchrotron radiation, intrabeam scattering and a noise from hardware, for example bunch by bunch feedback system [2]. These later two, which are not new subject, are related to essentials of the classical chaos dynamics.

## WEAK-STRONG MODEL

The beam-beam system including $N_{+}+N_{-}$particles characterized by trajectory in $6 \times\left(N_{+}+N_{-}\right)$dimensional phase space, where $N_{+}$and $N_{-}$are the number of particles in colliding two beams. We use so-called weak-strong model to treat this complex system. One beam is approximate to be a fixed charged distribution located at the collision point. The degree of freedom is now reduced to three coordinate and $s, 3+1=4$. The Hamiltonian is expressed by

$$
\begin{equation*}
H\left(x, p_{x}, y, p_{y}, z, p_{z} ; s\right)=\boldsymbol{\mu} \boldsymbol{J}_{0}+\delta_{P}(s) U\left(\boldsymbol{J}_{0}, \boldsymbol{\phi}_{0}\right) \tag{1}
\end{equation*}
$$

where $\boldsymbol{J}_{0}$ is invariant (action variable) defined in linearized system, $\delta_{P}(s)$ the periodic delta function and $U$ the potential of the beam-beam interaction.

If the system is solvable, $\boldsymbol{J}\left(\boldsymbol{J}_{0}, \boldsymbol{\phi}\right)$ is given by a canonical transformation so that Hamiltonian is only function of $\boldsymbol{J}$. A particle moves along a trajectory determined by $\boldsymbol{J}\left(\boldsymbol{J}_{0}, \boldsymbol{\phi}\right)=$ const.

For electron/positron ring, an equillibrum distribution of beam particles are determined by Fokker-Planck equation, which includes the radiation excitation and damping. The beam distribution for a solvable system is expressed by

$$
\begin{equation*}
\Psi(\boldsymbol{J}) \propto \exp \left(-\frac{J_{1}}{\varepsilon_{1}}-\frac{J_{2}}{\varepsilon_{2}}-\frac{J_{3}}{\varepsilon_{3}}\right) \tag{2}
\end{equation*}
$$

Since the new action $(\boldsymbol{J})$ is deviated from original action $J_{0}$, this distribution, which is distorted in the phase space, gives a kind of emittance growth, but the growth is limited.

Linear system is one of typical example of solvable system. Emittance growth does not exist in the linear system. System with one degree of freedom is another solvable system. Longitudinal dynamics can be one degree of freedom, since synchrotron tune is slow. We have an invariant $J(H)$, but have a kind of emittance growth, so-called potential distortion. However this emittance growth is considered as a redefinition of $J$ as shown in Eq.2. This type of emittance growth is not our subject.

Transverse dynamics have two degree of freedom at least, since Hamiltonian is a function of $s$. System with two or more degree of freedom is generally unsolvable For unsolvable system, invariant does not exist, in other words, the "invariant", which is defined in linearized system, is not invariant now. It is important how the "invariant" behaves in the nonlinear system, because emittance is given as an average of the "invariant" for beam particles. People know the number of degree of freedom is essential for the behavior of the "invariant".

In the system with two degree of freedom, motion of particle is represented with Poincare map in two dimensional phase space. Concentric circles, which are distorted by nonlinear force, is drown in two dimensional phase space for each initial amplitude (KAM surface). Some areas are not single solid curve, but are island and chaos. Anyway the "invariant" is bound in a region, though it is not an exact invariant. Therefore emittance growth is limited.

For three and four degree of freedom, particle motion is represented in four and six dimensional phase space, respectively. Concentric circles can not be drawn, if motion is mapped into two dimensional phase space: that is, the "invariant" are not bound in a region. Now the behavior of emittance is compete different from that of solvable or two
degree of freedom. Emittance, the average of the invariant for particles, has a diffusion nature: that is, emittance grows as a function of time like the solution of diffusion equation.

$$
\begin{equation*}
\varepsilon_{y}(t) / \varepsilon_{y}(0) \approx 1+D t \tag{3}
\end{equation*}
$$

where $D$ is diffusion rate. Needless to say, the increase of the emittance, which result from complex nonlinear dynamics, does not necessarily behave clearly linear, but should be monotonous.

Liapunov exponents [3] are used to characterize the chaotic behavior of nonlinear system quantitatively. We are interested in the evolution of beam size and luminosity, which is statistical behavior of the nonlinear chaotic system, but not interested in chaotic behavior of single particle. The diffusion rate can be a direct measure of the Chaotic behavior for our purpose.

The diffusion rate is estimated using a simulation based on the weak-strong model. Bassetti-Erskine formula [4] is used for the transverse beam-beam force and so-called synchro-beam mapping is used for the longitudinal [5]. The vertical beam size $\left\langle y^{2}\right\rangle-\langle y\rangle^{2}$ is calculated turn by turn and the diffusion rate is calculated by fitting a linear for the beam size variation. The radiation damping and excitation, which are included in usual weak-strong beam-beam simulations, are removed.

## DIFFUSION RATE

We first consider a two dimensional beam-beam system. The strong beam is approximated to be a thin charge distribution by bi-Gaussian with $\sigma_{x}$ and $\sigma_{y}$. Particles in the weak beam experiences a kick at $s=s^{*}+n L$, where $s^{*}$ is the collision point. This system is that with three degree of freedom system, since Hamiltonian is function of $x, p_{x}, y, p_{y}, s$.

The diffusion rate is calculated in tune space ( $\nu_{x}, \nu_{y}$ ) with step of 0.01 . The nominal beam-beam parameter is 0.14 . Figure 1 shows the diffusion rate in the tune space. The diffusion rate, which is comparable with or more than the damping rate, affects the beam-beam performance. For example of KEKB and DAFNE, which is B and $\phi$ factory machines, respectively, the damping rates of emittance are $2 / \tau=5 \times 10^{-4}$ and $2 \times 10^{-5}$ par turn, respectively. We mainly target accelerators with damping time of several 1000 turns like KEKB in this paper. For accelerators with slower damping time, simulation with a higher statistics to calculate the slower diffusion rate is required. The diffusion rate is negligible in wide tune area for KEKB in two dimensional model.

Figure 2 shows the diffusion rate in the tune space for three dimensional simulation with the bunch length, $\sigma_{z}=$ $\beta_{y}$. The diffusion rates are calculated for zero and finite crossing angle, $\sigma_{z} \theta / \sigma_{x}=1$ in plots (a) and (b), respectively.

We have an area with strong diffusion near $\left(\nu_{x}, \nu_{y}\right)=$ $(0.65,0.65)$. It is cross point of some resonances $3 \nu_{x}=$ $1,2 \nu_{y}+\nu_{x}=1$ and $3 \nu_{y}=1$. Each resonance line is


Figure 1: Diffusion rate of vertical beam size for twodimensional beam-beam system in the tune space $(0.5<$ $\nu_{x}<0.7,0.5<\nu_{y}<0.7$ ).
weak compare than that at the cross point. A area near $\nu_{x} \sim 0.5$ has a very low diffusion rate. For finite crossing angle, the region with lower diffusion rate than the damping rate is very narrow, is limited near $(0.51,0.55)$ which is just operating point of KEKB.


Figure 2: Diffusion rate of vertical beam size for threedimensional beam-beam system in tune space ( $0.5<\nu_{x}<$ $0.7,0.5<\nu_{y}<0.7$ ). Plots (a) and (b) are given for zero and finite crossing anle, respectively.

We now discuss the special characteristics of the operating point near a half integer of the horizontal tune. Everyone knows that the dynamic beta effect work strongly in the horizontal: that is, $\beta_{x} \rightarrow 0$. While the dynamic emittance effect makes increase $\varepsilon_{x} \rightarrow \infty$, with the result that $\sigma_{x} \rightarrow 0$ slowly.

Note that the diffusion rate of vertical beam size is focused now for $\nu_{x} \sim 0.5$. We image a particle motion which experiences the beam-beam force for $n u_{x}=0.5$. The horizontal coordinate $x$ of the particle is swapped $x \leftrightarrow-x$, with the result that the beam-beam force is
$F(x)=-F(-x)$. The motion of the particle is described in two-degree of freedom of $y$ and $s$. As is mentioned before, the diffusion for the system of two degree of freedom is limited. This means that very high luminosity can be achieved in this operating area. The same discussion is satisfied for $\nu_{x} \sim 0$. The figure and above discussion suggest as follows,

$$
\begin{equation*}
\lim _{x \rightarrow+0.5 \text { or }+0} D=0 \tag{4}
\end{equation*}
$$

Everyone knows that we have a strong beat of horizontal beta function and the momentum at the collision point diverge, $p_{x} \rightarrow \infty$, at the limit. Therefore an optimization of $\nu_{x}$ is necessary.

We discuss the motion near $\nu_{x}=0.5$ or 0 quantitatively. Sophisticated discussion is in Ref. [6]. One tur map of the particle is expressed by

$$
\begin{align*}
x & =\left(1-\frac{\mu_{x}^{2}}{2}\right) x+\beta_{x} \mu\left(1+\frac{\mu^{2}}{4}\right)  \tag{5}\\
p_{x} & =-\frac{\mu}{\beta} x+\left(1-\frac{\mu_{x}^{2}}{2}\right) p_{x}-F_{x}(x, y) \tag{6}
\end{align*}
$$

whre $\mu=2 \pi \nu_{x}$. The beam-beam force $F$ is expanded around $y=0$.

$$
\begin{equation*}
F_{x}(x, y)=F_{x}(x, 0)+\left.\frac{\partial F_{x}}{\partial y}\right|_{y=0} y+\left.\frac{1}{2} \frac{\partial^{2} F_{x}}{\partial^{2} y}\right|_{y=0} y^{2}+\ldots \tag{7}
\end{equation*}
$$

The first order term of the expansion is zero, and the second term gives a correction of $\sigma_{y} / \sigma_{x}$. Neglecting the correction term, the beam-beam force is only function of $x$. Then taken the limit $\mu \rightarrow 0$ is taken, the map is reduced to be a differential equation: that is, horizontal motion is decoupled and reduced to one degree of freedom. The differential equation is integrable and the solution is given as $x=x(s), p_{x}=p_{x}(s)$.

We confirm this result with a simulation. A particle trajectory is plotted in $x-p_{x}$ space for several horizontal tunes, $\nu_{x}=0.503,0.51,0.52$ and 0.54 , as shown in Figure 3. The particle has a vertical amplitude of $3 \sigma_{y}=0.03 \sigma_{x}$ at the initial condition. The trajectory nearer the half integer is expressed by a solid curve, while that apart from the half integer has a chaotic feature.

Once the horizontal motion is integrated, the vertical equation, which is represented by $x, y$ and $s$, is reduced to that for two degree of freedom, $x$ and $s$.

The longitudinal $z$ motion is solved as $z(s)$, since it is not strongly affected by the beam-beam force. The vertical motion is again reduced to two degree of freedom without crossing angle.

## CONCLUSION

We discussed the beam-beam limit caused by nonlinear diffusion. The number of degree of freedom is essential for the magnitude of the emittance growth. Operating point near $\nu_{x} \rightarrow+0.5$ and 0 gives the reduction of the degree of freedom, with the result that very high luminosity performance is expected.


Figure 3: Phase space plot in $x-p_{x} . y_{0}=2 \mu \mathrm{~m} \approx 3 \sigma_{y}$. plots (a), (b), (c) and (d) is given for $\nu_{x}=0.503,0.51,0.52$ and 0.54 , respectively.

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