NONLINEAR DAMPING OF INJECTION OSCILLATIONS

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Abstract

General analytical procedure for determination of the transverse feedback system parameters with a nonlinear regime of injection oscillations damping in circular accelerators and colliders is described. For this feedback loop the dependence between the kick value in the deflector and the beam deviation at the pick-up location is a nonlinear function. The beam dynamic nonlinear equation of the transverse coherent motion for deviation from the closed orbit has been solved using the approximation procedure based on the Krylov-Bogoliubov method for nonlinear differential equations. Approximate expressions for damping time and beam oscillation amplitude, including high order harmonics, are analysed for different nonlinear transfer functions.

1 INTRODUCTION

To damp the coherent transverse beam oscillations in synchrotrons, a transverse feedback system (TFS) is used. In this system the kicker (DK) corrects the beam angle according to the beam deviation from the closed orbit in the pick-up (PU) location at every turn. TFS has been widely used to suppress resistive wall instability and to provide a beam oscillation amplitude decrease after injection.

Electronics for signal processing in the feedback loop of most TFS is employed in order to obtain different dependencies $f(x)$ between the beam deviation $x[n, s_P]$ in PU and the kick $\Delta x'[n, s_K]$ in DK at the n-th turn:

$$\sqrt{\beta_P \beta_K}\, \Delta x'[n, s_K] \;=\; g\, f(x[n, s_P])\,. \qquad (1)$$

Here β_P and β_K are the transverse betatron amplitude functions in the PU and DK locations; g is the gain of the feedback loop. Usually power amplifiers with a linear characteristic are employed. Hence, a transfer function $f(x)$ of this feedback loop is a linear one (see Fig.1, dashed line).
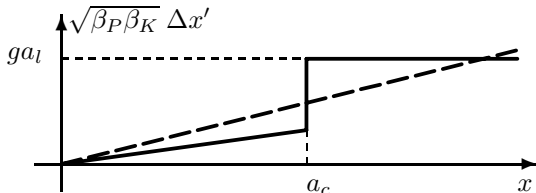

Figure 1: Dependence of the kick $\Delta x'$ on the beam deviation x for the linear (dashed line) and "bang-bang" (solid line) transfer functions

However, modern feedback loops of TFS include digital electronics (for example, filters and delays). Their transfer functions have a quasi-linear step character. Sometimes the

mode with limitation of a power for amplifiers is employed. So, the "bang-bang" mode of TFS operation in CERN SPS was realized to increase the efficiency of TFS [1]. Its transfer function consists of a linear part for small amplitude oscillations and a high fixed level gain for large oscillations (see Fig.1, solid line). The so-called "logical regime" was described in [2]. Its transfer function is a step function with two jumps. This nonlinear mode was tested in SPS [3]. The fast damping of initial oscillations was observed.

These experiments initiated some theoretical studies. A numerical simulation was used in [2] to estimate the damping time. It has been found that the amplitude of oscillations decreases linearly in time for some regimes.
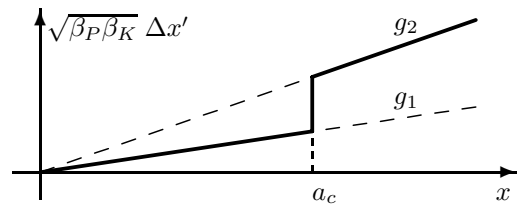

Figure 2: Dependence of the kick $\Delta x'$ on the beam deviation x for the transfer functions with a gain's jump from g_1 to g_2 (solid line)

Another example for designing a feedback loop with a special mode of operation is a system for damping of injection oscillations. For example, the feedback with a high level of a gain during a short period of operation is used in CERN SPS [4]. However, a feedback with a low gain level is used for damping of transverse instabilities. A nonlinear transfer function with a jump from high gain level to the small level (see Fig.1) can be used as a model to analize the process of switching off a high power TFS. This transfer function can be considered also as an example of a quasi-linear function.

So, modern electronics gives wide opportunities to design the feedback loops with different dependencies $f(x)$ between the beam deviation in PU and the kick in DK. As mentioned above, a transfer function $f(x)$ is a nonlinear one in these feedback systems.

As a rule, the numerical simulations are used to estimate the influence of the feedback parameters on beam dynamics when $f(x)$ is a nonlinear function. The analytical approach for nonlinear damping was developed in [5, 6] where the approximation procedure based on the Krylov–Bogoliubov method [7] for nonlinear differential equations was used. This approximation approach for nonlinear damping is generalized below. Approximate expressions for damping time and beam oscillation amplitude, including high order harmonics, are analysed for different

nonlinear transfer functions. All results are obtained for feedback description when instability is neglected.

2 THEORY

2.1 Basic Equation

The equation of the transverse coherent motion for the particle deviation from the closed orbit $x[n, s]$ at presence of a localized kick can be written at the n-th turn as

$$\left[\frac{d^2}{ds^2} + K(s)\right] x[n, s] = \Delta x'[n, s_K]\, \delta(s - s_K), \quad (2)$$

where $K(s)$ is a focusing strength and δ is Dirac's delta function. A localized kick is determined by $\Delta x'$ in (1).

It is shown in [5] that differential equation (2), when instability is neglected, can be transformed to the difference equation of the second order:

$$
\begin{aligned}
x[n+2, s_P] &- 2x[n+1, s_P]\cos\mu + x[n, s_P] \\
&= \sqrt{\beta_P \beta_K}\, \Delta x'[n+1, s_K]\sin(\mu - \eta) \\
&+ \sqrt{\beta_P \beta_K}\, \Delta x'[n, s_K]\sin\eta\,,
\end{aligned} \quad (3)
$$

where $\mu = 2\pi Q$ is a betatron phase advance per revolution in the transverse plane, Q is the number of unperturbed betatron oscillations per revolution, and η is the betatron phase advance from PU to DK.

Equations (3) and (1) are basic equations for studying beam dynamics with a nonlinear transfer function for a feedback loop. These equations are good for numerical calculations and convenient for analytical work.

2.2 Solution (First Approximation)

The gain g in (1) for feedback realized is a small value. Normally, $g \approx 0.01$ for instability damper systems and $g \approx 0.1$ for damping of injection errors. Since g is small, equation (3) is weakly nonlinear, and a number of perturbation methods is available to determine an approximate solution of this equation. It was demonstrated in [5, 6] that the Krylov–Bogoliubov method [7] can be used for solving the weakly nonlinear equation (3). This approach is used and generalized below.

When $g = 0$, the solution of (3) can be written as

$$
\begin{aligned}
x[n, s_P] &= a\cos(\mu n + \phi) = a\cos\psi_n\,, \quad (4)\\
\psi_n &= \mu n + \phi\,,
\end{aligned}
$$

where a and ϕ are constants. When $g \neq 0$, the solution of (3) can still be expressed in form (4), provided that a and ϕ are considered to be functions of n rather than constants. In accordance with the Krylov–Bogoliubov method, the solution of (3) can be written as a series of the form

$$x[n, s_P] = a_n\cos\psi_n + \sum_{m=1}^{\infty} g^m \xi_m(a_n, \psi_n)\,, \quad (5)$$

where ξ_i is unknown functions of full amplitude a_n and periodical functions of ψ_n. Functions ξ_i are small corrections

of the main harmonic $a_n\cos\psi_n$. The order of these corrections is given by small parameter g. The amplitude and phase are the functions of a_n. Hence, for their derivatives we can write:

$$
\begin{aligned}
\frac{da_n}{dn} &= gA_1(a_n) + g^2 A_2(a_n) + \dots\,, \quad (6)\\
\frac{d\psi_n}{dn} &= \mu + g\Phi_1(a_n) + g^2\Phi_2(a_n) + \dots\,. \quad (7)
\end{aligned}
$$

Functions ξ_m as the periodical functions of ψ_n can be expanded into the Fourier series:

$$\xi_m(a_n, \psi_n) = \sum_{k=-\infty}^{\infty} \widetilde{\xi}_{mk}(a_n)\exp(ik\psi_n)\,,$$

where $\widetilde{\xi}_{mk} = 0$ for $k = \pm 1$, because amplitude a_n is the full amplitude of the main (first) harmonic of oscillations.

For the left-hand side of (3) we expand all values into the Taylor series taking into account (6) and (7). The first approximation of these expansions is:

$$
\begin{aligned}
l.h.s. &\simeq gA_1(a_n)[\cos(\psi_n + 2\mu) - \cos\psi_n] \\
&+ g\Phi_1(a_n)a_n[\sin\psi_n - \sin(\psi_n + 2\mu)] \\
&+ g\xi_1(a_n, \psi_n) + g\xi_1(a_n, \psi_n + 2\mu) \\
&- 2g\cos\mu\,\xi_1(a_n, \psi_n + \mu)\,. \quad (8)
\end{aligned}
$$

For the first level of approximation, the right-hand side of (3) is determined by a zero level of approximation. Substituting for x from (5) and $\Delta x'$ from (1) into (3) yields

$$
\begin{aligned}
r.h.s. &= g\,f(a_n\cos\psi_n)\sin\eta \\
&+ g\,f(a_n\cos(\psi_n + \mu))\sin(\mu - \eta)\,. \quad (9)
\end{aligned}
$$

Equating coefficients of Fourier series in (8) and in (9) yields for main harmonic:

$$\frac{da_n}{dn} = -\frac{g}{2\pi}\int_0^{2\pi} f(a_n\cos\psi_n)\sin(\psi_n + \eta)\,d\psi_n\,; \quad (10)$$

$$\frac{d\psi_n}{dn} = \mu -$$

$$\frac{g}{2\pi a_n}\int_0^{2\pi} f(a_n\cos\psi_n)\cos(\psi_n + \eta)\,d\psi_n\,. \quad (11)$$

Equation (10) yields the amplitude damping rate per turn. The phase dependence on n for the beam transverse oscillations is determined by equation (11).

It is clear from (8) and (9) that the third and higher harmonics of oscillations can be excited. It depends on the transfer function. This is a typical effect for nonlinear systems. Equating coefficients of Fourier series in (8) and in (9) yields for higher harmonics ($|k| \neq \pm 1$):

$$\widetilde{\xi}_{mk}(a_n) = \frac{\sin(\mu - \eta) + \exp(ik\mu)\sin\eta}{2(\cos k\mu - \cos\mu)}\,\widetilde{f}_k(a_n), \quad (12)$$

where

$$\widetilde{f}_k(a_n) = \frac{1}{2\pi} \int\limits_0^{2\pi} f(a_n \cos \psi_n) \exp(-ik\psi_n) d\psi_n \; .$$

For TFS with a linear transfer function we have $f(x) = x$, where $x = a_n \cos \psi_n$ at zero level of approximation. Taking into account (5), (10), (11), and (12) we obtain the following solution:

$$x[n, s_P] \simeq a_0 \exp\left(-\frac{g}{2} n \sin \eta\right) \cos \psi_n \; ; \qquad (13)$$

$$\psi_n = \mu n + \phi_0 - \frac{g}{2} \cos \eta \; . \qquad (14)$$

where a_0 and ϕ_0 are constants depending on initial conditions. This solution coincides with the well known result (see, for example, [8]).

It is clear from (13) that if the phase advance η from the PU to the DK is equal to $\pi/2$ radians then the best damping is realised for TFS with the linear transfer function. In order to simplify all expressions, it will be supposed further that $\eta = \pi/2$.

3   RESULTS

3.1   "Bang-Bang" Damping

For TFS with a "bang-bang" transfer function we have (see Fig.1):

$$g\,f(x) = \begin{cases} g x_n & \text{when} \quad -a_c \le x_n \le a_c \, ; \\ g a_l & \text{when} \quad x_n > a_c \, ; \\ -g a_l & \text{when} \quad x_n < -a_c \, . \end{cases} \qquad (15)$$

Therefore, from (10) we have for $a_n < a_c$

$$\frac{da_n}{dn} \simeq -\frac{g}{2}\, a_n \; , \qquad (16)$$

and for $a_n > a_c$

$$\frac{da_n}{dn} \simeq -\frac{g}{2\pi}\left((4a_l - 2a_c)\sqrt{1 - \left(\frac{a_c}{a_n}\right)^2} + \left(\pi - 2\arccos\left(\frac{a_c}{a_n}\right)\right) a_n \right). \qquad (17)$$

For the phase of oscillations we get formula (14). Hence, to the first level of approximation, the frequency is not affected by the damping, while the amplitude decreases in accordance with dependence (17) or (16).

For $a_n \gg a_c$ we have from (17)

$$a_n \simeq a_0 - \frac{2}{\pi}\, g a_l n \; . \qquad (18)$$

Thus, to this level of approximation, large initial amplitudes decrease linearly with time. This formula (18) for amplitude dependence coincides with the result for amplitude solution of Coulomb damping. The linear amplitude decreasing with time was also obtained in [2] where a numerical simulation was used to estimate the damping time. Other results for the "bang-bang" regime are shown in [6].

3.2   Transfer Function with Gain's Jump

For TFS with gain's jump we have (see Fig.2):

$$g\,f(x) = \begin{cases} g_1 x_n & \text{when} \quad |x_n| \le a_c \, ; \\ g_2 x_n & \text{when} \quad |x_n| > a_c \, . \end{cases} \qquad (19)$$

Therefore, from (10) we have for $a_n < a_c$ the dependence (13) with a gain $g = g_1$, and for $a_n > a_c$

$$\frac{da_n}{dn} \simeq -\frac{g_1}{2}\, a_n - \frac{g_2 - g_1}{2}\, a_n \arccos\left(\frac{a_c}{a_n}\right) \; . \qquad (20)$$

For the phase of oscillations we get formula (14). Hence, to the first level of approximation, the frequency is not affected by the damping, while the amplitude decreases in accordance with dependence (13) or (20).

Thus, to this level of approximation, large initial amplitudes decrease faster with time than for TFS with a linear transfer function and a gain g_1 but slower than with a gain g_2.

4   CONCLUSION

The approaches demonstrated in [5, 6] have been developed and generalized in this paper for studying TFS with various nonlinear transfer functions. It gives analytical approximate solutions to calculate the damping time and other parameters of the particle motion. The approach can be used also to study beam dynamis for an extraction system.

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