A MODEL OF SELF-CONSISTENT TRANSVERSE DISTRIBUTION OF COLLIDING BUNCHES

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Abstract

The self-consistent beam-beam problem is formulated for the transverse distribution functions that may deviate from the Gaussian. The distribution function is constructed in terms of standard Gaussian distribution as a series of its derivatives. It gives the possibility to take into consideration the force from non-Gaussian opposite bunch and to estimate the threshold of the flip-flop effect.

1 INTRODUCTION

Various beam-beam effects have been observed on e^+e^- colliders and storage rings for many years. One of the interesting phenomena is a so-called flip-flop effect. For the behaviour of the bunches in meeting effects to be explained many models have been presented. All of them may be divided generally in two classes: studying the problem for a solid beam and taking into account particle distribution in the beam. The first kind of models takes the assumption that the bunch distribution is a standard Gaussian distribution and doesn't change itself due to the opposite bunch action. The second set of theories assumes the bunch distribution may deviate from the Gaussian shape, but often they suppose (for simplicity) the force from the opposite bunch can be considered as for unchanged Gaussian density. It is a good approximation for the problem to simplify, but if one supposes the possibility of varying the bunch density from the beam-beam effects one should consider the expression for the force in more complicated form. The aim of this paper is to present the self-consistent model taking into account the changes in the transverse distribution function of the bunch that gives the possibility to calculate the *exact* (in frames of this model) changed force from the opposite bunch and to study some aspects of the flip-flop effect.

2 NON-GAUSSIAN DISTRIBUTION FUNCTION

Let us begin from the standard 2D Gaussian distribution function for the density of round beam in transverse phase space:

$$\rho_0 = \frac{1}{2\pi s_1 s_1 \sqrt{1-r^2}} \times$$
$$\exp\left(-\frac{1}{2(1-r^2)}\left(\frac{x_1^2}{s_1^2}+\frac{2rx_1x_2}{s_1s_2}+\frac{x_2^2}{s_2^2}\right)\right), \quad (1)$$

where we made a notation: $x_1 = x$ for the coordinate (radius) and $x_2 = x'$ for the slope. In (1) s_1 and s_2 are respective RMS beam sizes in x_1 and x_2 directions, r is the usual correlation coefficient. We will assume for simplicity $r = 0$ in the following, it corresponds to choosing the Twiss parameter $\alpha = 0$ at the interaction point (IP).

Starting from the Gaussian distribution (1) now we are going to construct the non-Gaussian density function, which describes changed form of the bunch in phase space due to interaction with the opposite (also non-Gaussian) one. We want to approximate the distribution with 3 known second-order moments: (m_{11}, m_{12}, m_{22}). Both of the medians are assumed to be 0 because of the symmetry. Following [1] we will find the approximation as a truncated series of partial derivatives of ρ_0:

$$\rho = \rho_0 \cdot \left(1+\frac{C_3}{\rho_0}\frac{\partial^2\rho_0}{\partial x_1^2}+\frac{C_4}{\rho_0}\frac{\partial^2\rho_0}{\partial x_1\partial x_2}+\frac{C_5}{\rho_0}\frac{\partial^2\rho_0}{\partial x_2^2}\right). \quad (2)$$

The corresponding expressions for partial derivatives read:

$$\frac{\partial^2\rho_0}{\partial x_1^2} = \rho_0 \cdot \left(\frac{x_1^2}{s_1^4}-\frac{1}{s_1^2}\right), \quad \frac{\partial^2\rho_0}{\partial x_1\partial x_2} = \rho_0 \cdot \frac{x_1x_2}{s_1^2s_2^2},$$
$$\frac{\partial^2\rho_0}{\partial x_2^2} = \rho_0 \cdot \left(\frac{x_2^2}{s_2^4}-\frac{1}{s_2^2}\right). \quad (3)$$

Also we will assume that ρ in (2) represents normalized distribution.

Now we should express the set of coefficients (C_3, C_4, C_5) via assumed as known second-order moments (m_{11}, m_{12}, m_{22}). For this purpose we use the scheme of "orthogonalization", e. g. for the C_3:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \rho(x_1,x_2)\cdot\frac{1}{\rho_0}\frac{\partial^2\rho_0}{\partial x_1^2}\,dx_1\,dx_2 =$$
$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \rho_0(x_1,x_2)\cdot\frac{1}{\rho_0}\frac{\partial^2\rho_0}{\partial x_1^2} \times$$
$$\left(1+\frac{C_3}{\rho_0}\frac{\partial^2\rho_0}{\partial x_1^2}+\frac{C_4}{\rho_0}\frac{\partial^2\rho_0}{\partial x_1\partial x_2}+\frac{C_5}{\rho_0}\frac{\partial^2\rho_0}{\partial x_2^2}\right)dx_1dx_2. (4)$$

We use quotes around orthogonalization to emphasize that our base functions in (3) for ρ expansions are not orthogonal because of truncated form of (2). But this only means we assume high-order moments are small as compared with second-order moments and don't take them into consideration.

After simple calculation from (4) we obtain:

$$\frac{m_{11}}{s_1^4}-\frac{1}{s_1^2}=\frac{2C_3}{s_1^4}. \quad (5)$$

The same technique gives for C_4 and C_5:

$$\frac{m_{12}}{s_1^2 s_2^2} = \frac{C_4}{s_1^2 s_2^2}, \quad \frac{m_{22}}{s_2^4} - \frac{1}{s_2^2} = \frac{2C_5}{s_2^4}. \qquad (6)$$

Now we express (C_3, C_4, C_5) via (m_{11}, m_{12}, m_{22}) and finally obtain for the density function ρ:

$$\rho = \rho_0 \cdot \frac{1}{2} \cdot \left(4 - \frac{x_1^2}{s_1^2} - \frac{x_2^2}{s_2^2} + \frac{m_{11}(x_1^2 - s_1^2)}{s_1^4} + \right. \\ \left. + \frac{2m_{12}x_1x_2}{s_1^2 s_2^2} + \frac{m_{22}(x_2^2 - s_2^2)}{s_2^4}\right) \qquad (7)$$

with ρ_0 from (1).

So, we have the function, which approximate the distribution with known second-order moments. Put it in another way, (7) has beforehand known (m_{11}, m_{12}, m_{22}).

3 CALCULATING THE FORCE FROM NON-GAUSSIAN CHARGE DENSITY

The form of expression (7) allows us to calculate the force from the bunch having changed non-Gaussian density function. Usual method for this is applying directly the Gauss law to the bunch with known charge density to find the fields and then the force. If the bunch has the number of particles N and the charge density $\rho(r)$ then it produces the kick to the particle with the coordinate x_1 in counter moving bunch:

$$\Delta x_2 = -\frac{2Nr_e\sqrt{2\pi}}{s_1 x_1 \gamma} \int_0^{x_1} r\rho(r)\, dr. \qquad (8)$$

Here r_e is a classical electron radius, γ is a Lorentz factor.

Performing the integration in our case of the density (7) we obtain the kick.

For simplicity in the following it is convenient express the last formula via the value of beam-beam parameter for round beam: $\xi = Nr_e\beta^*/4\pi\gamma s_1^2$, where β^* is the unperturbed value of the β-function at the IP:

$$\Delta x_2 = -\frac{4\pi\xi}{s_1^2 x_1 \beta^*}\Big[s_1^2(m_{11} + s_1^2) - \\ - e^{-\frac{x_1^2}{2s_1^2}}\Big(s_1^2(m_{11} + s_1^2) + x_1^2(m_{11} - s_1^2)\Big)\Big]. \qquad (9)$$

Fig. 1 shows the dependence of the kick on the value of m_{11} for non-Gaussian charge distribution in the bunch. All the curves are normalized on the maximal value of the standard Gaussian density function.

4 THE FLIP-FLOP EFFECT

The expression for the force from non-Gaussian bunch obtained allows us now to apply it to the studying in our model the flip-flop effect. As it is known this phenomenon appears in such a way that the sizes of colliding bunches may differ from the normal once in many times. Now we construct the self-consistent model of bunch distribution including betatron oscillations, radiation effects and nonlinear kick from the opposite bunch.

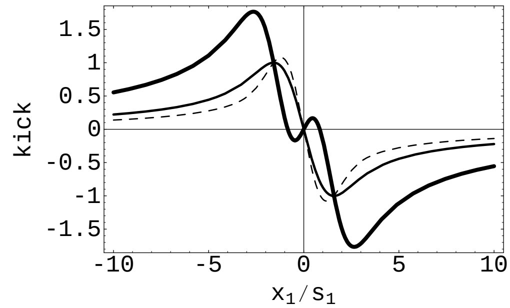

Figure 1: The dependence of the kick (9) on the value of relative coordinate deviation x_1/s_1. Thin curve is the dependence in the case of standard Gaussian density, thick and dashed lines represent the case when the size of the acting bunch is two times as large (and as small) as the normal size respectively

At first, we should transform the bunch distribution through the nonlinear kick. We apply the transformation:

$$x_1 \to x_1, \quad x_2 \to x_2 + \Delta x_2$$

with Δx_2 in the form (9) and with a substitution $m_{11} \to M_{11}$ (the opposite bunch) to the distribution function (7). Then we calculate the new values of the second-order moments of the distribution obtained. For this calculation to simplify it is convenient to use a *special order* of integration: at first integrate over the x_2 and then — over the x_1. In this case at first integration we may consider the complicated (exponential) adding in x_2 as a “shift” of the origin and perform the integration. After the integration we get the new values of the second-order moments and construct the new distribution function (7) with these new values of (m_{11}, m_{12}, m_{22}).

Then we apply to this distribution the standard map: betatron oscillation and the radiation effects in the form of [2]. Now we have the distribution function at the end of one revolution, hence we may calculate the final values of the second-order moments and construct the self-consistent system. Because of very complicated form all the following analysis and solving has been made numerically.

The main conclusion from the solving this system obtained is the flip-flop for the round beam may occur only if the working point ν is near 0.5 (half-integer resonance). In this case the flip-flop threshold is about $\xi \sim 0.1$ with λ is around 0.95. For more intensive damping ($\lambda \simeq 0.85$) the flip-flop effect takes place beginning from the very small values of ξ. The two equal solutions exist before ξ exceeds the value about 0.01, if ξ is large than this value no solutions exist. At the working point $\nu = 0.1$ there are some ranges of only equal solutions existence. We also found that there is no serious dependence of solutions on the values of the β-function at the IP (β^*), except the flip-flop at

$\nu = 0.4$ appears at higher values of ξ when we increased the β^*. The behaviour of the density function near the flip-flop limit is shown in the Fig. 2.

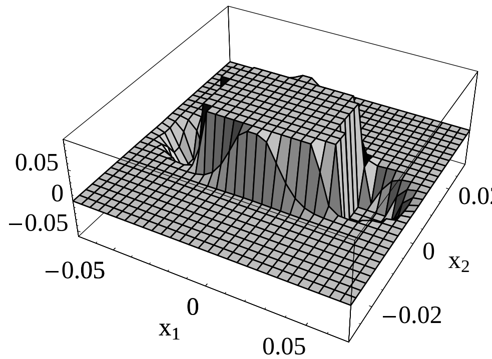

Figure 2: The distribution function (7) near the flip-flop threshold, when the areas of the negative density appear

5 CONCLUSION

Of course, our model has the same disadvantage as the model presented in [1] — at some values of parameters the density of phase space distribution (7) may lose its positive definition and become negative. It is concerned with the truncated form of the series expansion (2). We observed that when parameters are far enough from the flip-flop threshold there are no ranges of negative density, or they are small. When the parameters are near the critical values the area of negative density is increased, but the positive central part has the size of about $(8 \div 10)\sqrt{m_{11}} \times (8 \div 10)\sqrt{m_{22}}$.

In our paper we presented the self-consistent model of transverse distribution of the non-Gaussian colliding bunches based on the series expansion of the standard Gaussian distribution. This approach allows one to calculate the force for the non-Gaussian bunch instead of using the Gaussian expression for the force from opposite bunch as in [1]. Our technique leads to more accurate account of the beam-beam effects.

REFERENCES

[1] K. Hirata Beyond Gaussian Approximation for Beam-Beam Interaction CERN/LEP-TH/88-56.

[2] K. Hirata and F. Ruggiero. Treatment of Radiation in Electron Storage Rings. LEP Note 611 (1988).

A POSITIVENESS OF THE DENSITY FUNCTION

In this appendix we will discuss some criterions for the density function to be positive. The expression (7) may be rewritten in such a way:

$$\rho = \exp\left(-\frac{x_1^2}{2s_1^2} - \frac{x_2^2}{2s_2^2}\right) \cdot \frac{1}{4\pi s_1^5 s_2^5} \cdot p(x_1, x_2), \quad (10)$$

where $p(x_1, x_2)$ is a polynomial of the two variables and reads:

$$p(x_1, x_2) = m_{22} s_1^4 (x_2^2 - s_2^2) - s_2^2 [s_2^2(-4s_1^4 + s_1^2 x_1^2 + \\ + m_{11}(s_1^2 - x_1^2)) - 2m_{12} s_1^2 x_1 x_2 + s_1^4 x_2^2]. \quad (11)$$

For the density ρ to be positive we need satisfying very simple conditions: the polynomial $p(x_1, x_2) > 0$ for $x_1 = 0$, $x_2 = 0$ and it must be positive defined polynomial, i. e. the eigenvalues of a matrix of its second-order derivatives both are greater than zero. The requirement of $p(x_1, x_2) > 0$ at the origin leads to:

$$\frac{m_{11}}{s_1^2} + \frac{m_{22}}{s_2^2} < 4. \quad (12)$$

And from the condition on the eigenvalues to be positive we have:

$$s_2^4(m_{11} - s_1^2) + s_1^4(m_{22} - s_2^2) > 0 \quad (13)$$

$$(m_{11} - s_1^2)(m_{22} - s_2^2) - m_{12}^2 > 0. \quad (14)$$

The inequalities (12) and (13) define two lines on the plane (m_{11}, m_{22}) and the condition (14) gives an additional constraint. These describe the area of allowed values of the second-order moments and this area corresponds to the density function must be positive defined by the physical sense. The situation is presented schematically in Fig. 3.

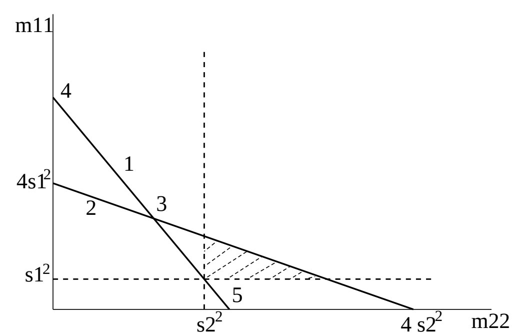

Figure 3: The area (hatched) of the allowed values of the second-order moments for the density function to be positive. The curve 1 corresponds to the inequality (13), the curve 2 illustrates the condition (12). The numbered points have coordinates: 3 — $(s_2^2(s_1^2 - 3s_2^2)/(s_1^2 - s_2^2), s_1^2(3s_1^2 - s_2^2)/(s_1^2 - s_2^2))$, 4 — $(0, s_1^2(1 + s_1^2/s_2^2))$, 5 — $(s_2^2(1 + s_2^2/s_1^2), 0)$. Dashed lines are drawn with respect to the inequality (14)

From this picture it is clearly seen that for the density ρ in our model to be positive the values of (m_{11}, m_{22}) should be greater than their nominal values (s_1^2, s_2^2). But in addition they should not be very big, i. e. in the flip-flop state (when the sizes are different in many times) the density function has negative areas in most cases as we saw. In other words the consistent bunch size should has slow increasing with the intensity of the opposite bunch to avoid the unwanted flip-flop state and this behaviour is in agreement with requirement of a positiveness of the density function as we have seen in numerical studying of our model.