

# LOCAL ANALYSIS OF NONLINEAR RMS ENVELOPE DYNAMICS

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## Abstract

We present applications of variational – wavelet approach to nonlinear (rational) rms envelope dynamics. We have the solution as a multiresolution (multiscales) expansion in the base of compactly supported wavelet basis.

## 1 INTRODUCTION

In this paper we consider the applications of a new numerical-analytical technique which is based on the methods of local nonlinear harmonic analysis or wavelet analysis to the nonlinear root-mean-square (rms) envelope dynamics [1]. Such approach may be useful in all models in which it is possible and reasonable to reduce all complicated problems related with statistical distributions to the problems described by systems of nonlinear ordinary/partial differential equations. In this paper we consider an approach based on the second moments of the distribution functions for the calculation of evolution of rms envelope of a beam. The rms envelope equations are the most useful for analysis of the beam self–forces (space–charge) effects and also allow to consider both transverse and longitudinal dynamics of space-charge-dominated relativistic high–brightness axisymmetric/asymmetric beams, which under short laser pulse–driven radio-frequency photoinjectors have fast transition from nonrelativistic to relativistic regime [2]. From the formal point of view we may consider rms envelope equations after straightforward transformations to standard Cauchy form as a system of nonlinear differential equations which are not more than rational (in dynamical variables). Because of rational type of nonlinearities we need to consider some extension of our results from [3]-[10], which are based on application of wavelet analysis technique to variational formulation of initial nonlinear problems.

Wavelet analysis is a relatively novel set of mathematical methods, which gives us a possibility to work with well-localized bases in functional spaces and give for the general type of operators (differential, integral, pseudodifferential) in such bases the maximum sparse forms. Our approach in this paper is based on the generalization [11] of variational-wavelet approach from [3]-[10], which allows us to consider not only polynomial but rational type of nonlinearities.

In part 2 we describe the different forms of rms equations. In part 3 we present explicit analytical construction for solutions of rms equations from part 2, which are based on our variational formulation of initial dynamical problems and on multiresolution representation [11]. We

give explicit representation for all dynamical variables in the base of compactly supported wavelets. Our solutions are parametrized by solutions of a number of reduced algebraical problems from which one is nonlinear with the same degree of nonlinearity and the rest are the linear problems which correspond to particular method of calculation of scalar products of functions from wavelet bases and their derivatives. In part 4 we consider results of numerical calculations.

## 2 RMS EQUATIONS

Below we consider a number of different forms of RMS envelope equations, which are from the formal point of view not more than nonlinear differential equations with rational nonlinearities and variable coefficients. Let  $f(x_1, x_2)$  be the distribution function which gives full information about noninteracting ensemble of beam particles regarding to trace space or transverse phase coordinates  $(x_1, x_2)$ . Then we may extract the first nontrivial bit of ‘dynamical information’ from the second moments

$$\begin{aligned}\sigma_{x_1}^2 &= \langle x_1^2 \rangle = \int \int x_1^2 f(x_1, x_2) dx_1 dx_2 \\ \sigma_{x_2}^2 &= \langle x_2^2 \rangle = \int \int x_2^2 f(x_1, x_2) dx_1 dx_2 \\ \sigma_{x_1 x_2}^2 &= \langle x_1 x_2 \rangle = \int \int x_1 x_2 f(x_1, x_2) dx_1 dx_2\end{aligned}\quad (1)$$

RMS emittance ellipse is given by  $\varepsilon_{x,rms}^2 = \langle x_1^2 \rangle \langle x_2^2 \rangle - \langle x_1 x_2 \rangle^2$ . Expressions for twiss parameters are also based on the second moments.

We will consider the following particular cases of rms envelope equations, which described evolution of the moments (1) ([1],[2] for full designation): for asymmetric beams we have the system of two envelope equations of the second order for  $\sigma_{x_1}$  and  $\sigma_{x_2}$ :

$$\begin{aligned}\sigma_{x_1}'' + \sigma_{x_1}' \frac{\gamma'}{\gamma} + \Omega_{x_1}^2 \left( \frac{\gamma'}{\gamma} \right)^2 \sigma_{x_1} &= \\ I / (I_0 (\sigma_{x_1} + \sigma_{x_2}) \gamma^3) + \varepsilon_{n x_1}^2 / \sigma_{x_1}^3 \gamma^2, \\ \sigma_{x_2}'' + \sigma_{x_2}' \frac{\gamma'}{\gamma} + \Omega_{x_2}^2 \left( \frac{\gamma'}{\gamma} \right)^2 \sigma_{x_2} &= \\ I / (I_0 (\sigma_{x_1} + \sigma_{x_2}) \gamma^3) + \varepsilon_{n x_2}^2 / \sigma_{x_2}^3 \gamma^2\end{aligned}\quad (2)$$

The envelope equation for an axisymmetric beam is a particular case of preceding equations.

Also we have related Lawson’s equation for evolution of the rms envelope in the paraxial limit, which governs evolution of cylindrical symmetric envelope under external

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linear focusing channel of strenghts  $K_r$ :

$$\sigma'' + \sigma' \left( \frac{\gamma'}{\beta^2 \gamma} \right) + K_r \sigma = \frac{k_s}{\sigma \beta^3 \gamma^3} + \frac{\varepsilon_n^2}{\sigma^3 \beta^2 \gamma^2}, \quad (3)$$

where  $K_r \equiv -F_r/r\beta^2\gamma mc^2$ ,  $\beta \equiv v_b/c = \sqrt{1-\gamma^{-2}}$

After transformations to Cauchy form we can see that all this equations from the formal point of view are not more than ordinary differential equations with rational nonlinearities and variable coefficients (also, we may consider regimes in which  $\gamma, \gamma'$  are not fixed functions/constants but satisfy some additional differential constraint/equation, but this case does not change our general approach).

### 3 RATIONAL DYNAMICS

The first main part of our consideration is some variational approach to this problem, which reduces initial problem to the problem of solution of functional equations at the first stage and some algebraical problems at the second stage. We have the solution in a compactly supported wavelet basis. An example of such type of basis is demonstrated on Fig. 1. Multiresolution representation is the second main

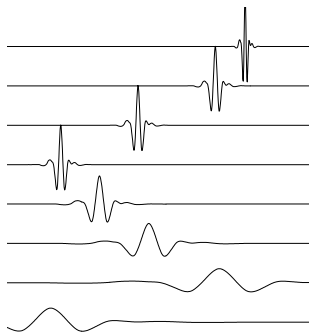


Figure 1: Wavelets at different scales and locations.

part of our construction. The solution is parameterized by solutions of two reduced algebraical problems, one is nonlinear and the second are some linear problems, which are obtained from one of the standard wavelet constructions: the method of Connection Coefficients (CC) or Stationary Subdivision Schemes (SSS).

So, our variational-multiresolution approach [11] gives us possibility to construct explicit numerical-analytical solution for the following systems of nonlinear differential equations

$$\dot{z} = R(z, t) \quad \text{or} \quad Q(z, t)\dot{z} = P(z, t), \quad (4)$$

where  $z(t) = (z_1(t), \dots, z_n(t))$  is the vector of dynamical variables  $z_i(t)$ ,

$R(z, t)$  is not more than rational function of  $z$ ,

$P(z, t), Q(z, t)$  are not more than polynomial functions of  $z$  and  $P, Q, R$  have arbitrary dependence of time.

The solution has the following form

$$z(t) = z_N^{slow}(t) + \sum_{j \geq N} z_j(\omega_j t), \quad \omega_j \sim 2^j \quad (5)$$

which corresponds to the full multiresolution expansion in all time scales. Formula (5) gives us expansion into a slow part  $z_N^{slow}$  and fast oscillating parts for arbitrary  $N$ . So, we may move from coarse scales of resolution to the finest one for obtaining more detailed information about our dynamical process. The first term in the RHS of representation (5) corresponds on the global level of function space decomposition to resolution space and the second one to detail space. In this way we give contribution to our full solution from each scale of resolution or each time scale. The same is correct for the contribution to power spectral density (energy spectrum): we can take into account contributions from each level/scale of resolution.

So, we have the solution of the initial nonlinear (rational) problem in the form

$$z_i(t) = z_i(0) + \sum_{k=1}^N \lambda_i^k Z_k(t), \quad (6)$$

where coefficients  $\lambda_i^k$  are roots of the corresponding reduced algebraical (polynomial) problem [11]. Consequently, we have a parametrization of solution of initial problem by solution of reduced algebraical problem.

So, the obtained solutions are given in the form (6), where  $Z_k(t)$  are basis functions and  $\lambda_i^k$  are roots of reduced system of equations. In our case  $Z_k(t)$  are obtained via multiresolution expansions and represented by compactly supported wavelets and  $\lambda_i^k$  are the roots of reduced polynomial system with coefficients, which are given by CC or SSS constructions.

Each  $Z_j(t)$  is a representative of corresponding multiresolution subspace  $V_j$ , which is a member of the sequence of increasing closed subspaces  $V_j$ :

$$\dots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \dots \quad (7)$$

The basis in each  $V_j$  is

$$\varphi_{jl}(x) = 2^{j/2} \varphi(2^j x - \ell) \quad (8)$$

where indices  $\ell, j$  represent translations and scaling respectively or action of underlying affine group which act as a "microscope" and allow us to construct corresponding solution with needed level of resolution.

It should be noted that such representations (5),(6) for solutions of equations (2),(3) give the best possible localization properties in corresponding phase space. This is especially important because our dynamical variables corresponds to moments of ensemble of beam particles.

### 4 NUMERICAL CALCULATIONS

In this part we consider numerical illustrations of previous analytical approach. Our numerical calculations are based on compactly supported Daubechies wavelets and related wavelet families. On Fig. 2 we present according to formulae (5),(6) contributions to approximation of our dynamical evolution (top row on the Fig. 3) starting from the coarse

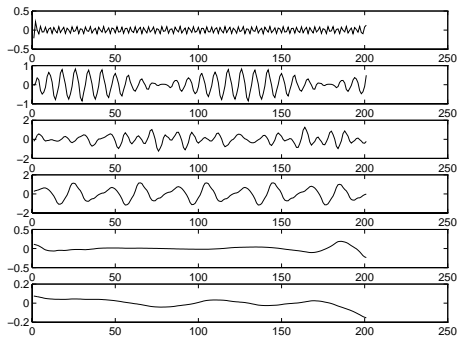


Figure 2: Contributions to approximation: from scale  $2^1$  to  $2^5$ .

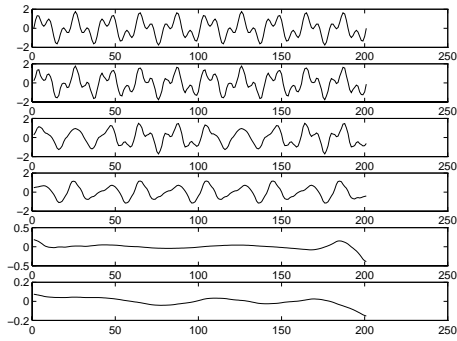


Figure 3: Approximations: from scale  $2^1$  to  $2^5$ .

approximation, corresponding to scale  $2^0$  (bottom row) to the finest one corresponding to the scales from  $2^1$  to  $2^5$  or from slow to fast components (5 frequencies) as details for approximation. Then on Fig. 3, from bottom to top, we demonstrate the summation of contributions from corresponding levels of resolution given on Fig. 2 and as result we restore via 5 scales (frequencies) approximation our dynamical process (top row on Fig. 3).

We also produce the same decomposition/approximation on the level of power spectral density (Fig. 4). It should be noted that complexity of such algorithms are minimal regarding other possible. Of course, we may use different multiresolution analysis schemes, which are based on different families of generating wavelets and apply such schemes of numerical-analytical calculations to any dynamical process which may be described by systems of ordinary/partial differential equations with rational nonlinearities [11].

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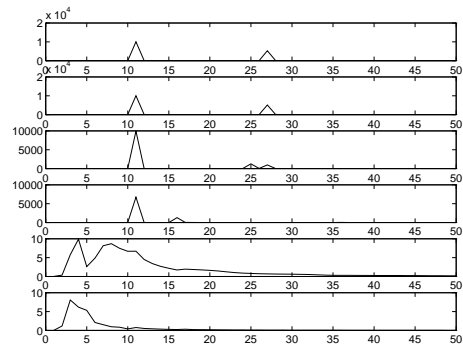


Figure 4: Power spectral density: from scale  $2^1$  to  $2^5$ .

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