LIE METHOD ANALYSIS FOR THE NONLINEAR TRANSPORT OF INTENSE BEAM WITH K-V DISTRIBUTION

Lü Jian-Qin
Institute of Heavy Ion Physics, Peking University, Beijing 100871, China

Abstract

Nonlinear transport of intense charged particle beams is analyzed with Lie algebraic methods. The particle distribution in six-dimensional phase spaces is of K-V type. The analysis is performed for magnetic quadrupoles, and it is similar for dipoles, sextupoles and other optical elements.

1 Introduction

When the particle energy is low and the beam current is high, the space charge force of the beams can not be ignored. In this case, accuracy calculations for the particle trajectories are very complicated. So, linear approximation is usually adopted. However, in the beam transport experiments of intense beams, beam hallo can be observed obviously, even the hallo beams are cut off with aperture lilts, on the following beam lines, beam hallo still appears. That is because of nonlinear effects of the beam optics elements, especially for the intense beams. In the intense accelerators, such as medical proton linear accelerators, Accelerator Driven System Nuclear Power and so on, the nonlinear transport of intense beams should be taken into account, so that high beam transmission can be obtained.

There are two ways to calculate nonlinear transport for the intense beams: numerical methods (That is solving fields and calculating trajectories) and analytical methods. The former methods are usually used for short beam transport systems (say, ion attracting systems in the front of ion sources). Because of large memory equerry of numerical calculations, analytical approach is convenient for the very long beam line calculations.

Lie algebraic methods[1] provide a good tool to study nonlinear transport of intense beams. The key problem is how to express the electric potentials of the beams. Because different particle phase space distributions have different potentials, and they will evolve with the particle motions. So, it is a very difficult problem to calculate electric potentials of the beams. However, in the case of K-V distributions, the electric potentials of the beams can be calculated easily. In this paper, we present the nonlinear transport of intense beams in quadrupole magnets analyzed with Lie algebraic methods.

2 Hamiltonian and its expansion[2]

In the Cartesian coordinates (x,y,z), the Hamiltonian of a particle with time t as independent variable is

$$H_t = [m_0^2 c^4 + c^2(p_x - qA_x)^2 + c^2(p_y - qA_y)^2 + c^2(p_z - qA_z)^2 +]^{\frac{1}{2}} + q\varphi \qquad (1)$$

where m_0 is the particle rest energy, q is the charge, p_x, p_y and p_z is the x, y, and z component of the particle momentum, A_x, A_y and A_z is the x, y, and z component of the magnetic vector potential, ϕ is the electric potential, c is the light velocity. Here, the canonical variables are η=(x, y, z, p_x, p_y, p_z).

Introduce variable $p_t = -H_t$(x，y，z，p_x，p_y，p_z；t)，solve p_z from p_t , one obtains：

$$K = -p_z = -[(p_t + q\phi)^2 / c^2 - m_0^2 c^2 - (p_x - qA_x)^2 - (p_y - qA_y)^2]^{\frac{1}{2}} - qA_z \qquad (2)$$

Define new canonical variables ζ=（x, y,τ,x' , y', p_τ）:

$$x = x,\quad y = y,\quad \tau = T - z/\beta_0,$$
$$x' = px/p_0,\quad y' = py/p_0,\quad p_\tau = p_T - p_T^0. \qquad (3)$$

where T =ct ，$\beta_0 = c/v_0$（v_0 is the velocity of reference particle）; p_0 is the momentum of reference particle; $p_T = p_t/(p_0 c)$; p_T^0 is the value of p_T for reference particle。

Under the transformation expressed by eq.(3), the new Hamiltonian is

$$H = -[(p_\tau + p_T^0 + \frac{q\phi}{p_0 c})^2 - \frac{1}{\beta_0^2\gamma^2} - (x' - qA_x/p_0)^2 - (y' - qA_y/p_0)^2]^{\frac{1}{2}} - qA_z/p_0 - (p_\tau + p_T^0)/\beta_0 \qquad (4)$$

For the magnetic quadrupoles, $\mathbf{A} = \frac{G}{2}(y^2 - x^2)\mathbf{e}_z$, and the electric potential excited by the charged particle beams in the case of K-V distribution is

$$\phi = -\frac{3IT_{rf}}{8\pi\varepsilon_0\gamma_0 XYZ}(\mu_x x^2 + \mu_y y^2 + \mu_z \beta_0^2 \tau^2) \qquad (5)$$

where I is the beam current of the beam bundles; T_{rf} is the beam repetition period; X, Y and Z are the pulsed beam dimensions; z_r is the longitudinal position of the arbitrary particle relative to what of the reference particle; μ_x, μ_y and μ_z are the beam shape factors of the bundles, expressed as

$$\mu_x = \frac{XYZ\gamma}{2}\int_0^\infty \frac{d\xi}{(x^2+\xi)\sqrt{(x^2+\xi)(y^2+\xi)(z^2\gamma^2+\xi)}}$$

$$\mu_y = \frac{XYZ\gamma}{2}\int_0^\infty \frac{d\xi}{(y^2+\xi)\sqrt{(x^2+\xi)(y^2+\xi)(z^2\gamma^2+\xi)}} \qquad (6)$$

$$\mu_z = \frac{XYZ\gamma}{2}\int_0^\infty \frac{d\xi}{(z^2\gamma^2+\xi)\sqrt{(x^2+\xi)(y^2+\xi)(z^2\gamma^2+\xi)}}$$

Substitute eq.(5) into eq.(4) , one obtains

$$H = -\{[(p_\tau + p_T^0 - Q(\mu_x x^2 + \mu_y y^2 + \mu_z \beta_0^2 \tau^2)]^2 - \frac{1}{\beta_0^2\gamma^2} - x'^2 - y'^2]\}^{\frac{1}{2}} + \frac{qG}{2p_0}(x^2+y^2) - (p_\tau + p_T^0)/\beta_0 \qquad (7)$$

where

$$Q = \frac{3IT_{rf}}{8\pi\varepsilon_0\gamma_0 p_0 cXYZ} \qquad (8)$$

Expand the Hamiltonian (7) about the equilibrium orbit, we have

$$\mathrm{H} = \sum_{n=0}^{\infty} H_n \qquad (9)$$

where

$$\mathrm{H}_0 = \frac{1}{\beta_0^2\gamma_0^2},$$

$$\mathrm{H}_1 = 0$$

$$\mathrm{H}_2 = (\frac{qG}{2p_0} - Q\mu_x/\beta_0)x^2 - (\frac{qG}{2p_0} + Q\mu_x/\beta_0)y^2 + \frac{1}{2}(x'^2 + y'^2) - Q\mu_z\beta_0\tau^2 + \frac{1}{2\beta_0^2\gamma_0^2}p_\tau^2$$

$$\mathrm{H}_3 = -\frac{Q\mu_x}{\beta_0^2\gamma_0^2}x^2 p_\tau - \frac{Q\mu_y}{\beta_0^2\gamma_0^2}y^2 p_\tau + \frac{1}{2\beta_0}(x'^2 p_\tau + y'^2 p_\tau) - \frac{Q\mu_z}{\gamma_0^2}\tau^2 p_\tau + \frac{1}{2\beta_0^3\gamma_0^2}p_\tau^3$$

..... (10)

3 First order approximation

The linear map $\mathbf{M}_2$ is expressed as

$$\mathbf{M}_2 = \exp(-:f_2:) \qquad (11)$$

where, $:f_2:$ is Lie operator, when acting on another function, it perform Poisson bracket operation, and

$$f_2 = -\ell H_2 \quad (\ell \text{ is the length of quadrupoles}) \qquad (12)$$

Let the subscript “1”expresses the first order terms of the map, and $\mathbf{M}_2$ act on the components of the canonical variable ζ, one obtains the first order approximation solutions of the particle trajectories, expressed in matrix form, they are

$$\begin{bmatrix} x_1 \\ x_1' \\ y_1 \\ y_1' \\ \tau_1 \\ p_{\tau 1} \end{bmatrix} = \begin{bmatrix} \cos(k_x\ell) & \frac{1}{k_x}\sin(k_x\ell) & 0 & 0 & 0 & 0 \\ -k_x\sin(k_x\ell) & \cos(k_x\ell) & 0 & 0 & 0 & 0 \\ 0 & 0 & \cosh(k_y\ell) & \frac{1}{k_y}\sinh(k_y\ell) & 0 & 0 \\ 0 & 0 & k_y\sinh(k_y\ell) & \cosh(k_y\ell) & 0 & 0 \\ 0 & 0 & 0 & 0 & \cosh(k_z\ell) & \frac{\sinh(k_z\ell)}{k_z\beta_0^2\gamma_0^2} \\ 0 & 0 & 0 & 0 & k_z\beta_0^2\gamma_0^2\sinh(k_z\ell) & \cosh(k_z\ell) \end{bmatrix} \cdot \begin{bmatrix} x \\ x' \\ y \\ y' \\ \tau \\ p_\tau \end{bmatrix} \quad (13)$$

where $k_x^2 = -\frac{qG}{p_0} + 2Q\mu_x/\beta_0$, $k_y^2 = \frac{qG}{p_0} + 2Q\mu_y/\beta_0$,

$$k_z^2 = \frac{2Q\mu_z}{\beta_0\gamma_0^2} \quad (14)$$

4 Second order approximation

The second order map $\mathbf{M}_3$ can be expressed as $\mathbf{M}_3$=:f$_3$:, where

$$f_3 = -\int_0^\ell h_3^{\text{int}}(\zeta, z_1)\,dz_1 = -\int_0^\ell \mathbf{M}_2 H_3(\zeta, z_1)\,dz_1 \quad (15)$$

Let the map $\mathbf{M}_3$ act on the linear solution $\zeta_1 = (x,\ x',\ y,\ y',\ \tau,\ p'_\tau,)$, one obtains the second order solutions ζ_2 (the subscript "2" expresses second order) of the map. The results are listed as the following

$$\begin{aligned} x_2 = {} & \frac{x\tau}{4k_x^2+k_z^2}\left\{\frac{2Q\mu_x}{k_x}\left[-(2k_x^2+k_z^2)\sin(k_x\ell)+\right.\right. \\ & \left. 2k_x^2\sin(k_x\ell)\cosh(k_z\ell)+k_xk_z\cos(k_x\ell)\sinh(k_z\ell)\right]+ \\ & k_x\beta_0\gamma_0^2\left[2k_x^2\sin(k_x\ell)-(2k_x^2+k_z^2)\sin(k_x\ell)\cosh(k_z\ell)+\right. \\ & \left.\left. k_xk_z\cos(k_x\ell)\sinh(k_z\ell)\right]\right\}+ \\ & \frac{x\tau'}{(4k_x^2+k_z^2)k_z}\left\{\frac{2Q\mu_x}{\beta_0^2\gamma_0^2}\left[k_z\cos(k_x\ell)(-1+\cosh(k_z\ell))+\right.\right. \\ & \left.2k_x\sin(k_x\ell)\sinh(k_z\ell)\right]+ \\ & \frac{k_x}{\beta_0}\left[k_xk_z\cos(k_x\ell)(-1+\cosh(k_z\ell))-\right. \\ & \left.\left.(2k_x^2+k_z^2)\sin(k_x\ell)\sinh(k_z\ell)\right]\right\}+ \\ & \frac{x'\tau}{4k_x^2+k_z^2}\left\{\frac{2Q\mu_x}{k_x}\left[2k_x\cos(k_x\ell)(1-\cosh(k_z\ell))+\right.\right. \\ & \left.k_z\sin(k_x\ell)\sinh(k_z\ell)\right]+ \\ & \beta_0\gamma_0^2\left[(2k_x^2+k_z^2)\cos(k_x\ell)(-1+\cosh(k_z\ell))+\right\}+ \\ & \left.\left.k_xk_z\sin(k_x\ell)\sinh(k_z\ell)\right]\right\}+ \\ & \frac{x'\tau'}{(4k_x^2+k_z^2)k_z}\left\{\frac{2Q\mu_x}{k_x\beta_0^2\gamma_0^2}\left[k_z\sin(k_x\ell)(1+\cosh(k_z\ell))-\right.\right. \\ & \left.2k_x\cos(k_x\ell)\sinh(k_z\ell)\right]+ \\ & \frac{1}{\beta_0}\left[k_xk_z\sin(k_x\ell)(1+\cosh(k_z\ell))+\right. \\ & \left.\left.(2k_x^2+k_z^2)\cos(k_x\ell)\sinh(k_z\ell)\right]\right\} \end{aligned} \quad (16)$$

Because the paper is limited up to three pages, the second terms of x_2', y_2, y_2', τ_2, *and* $p_{\tau 2}$ are not listed here.

5 Discussions

It is a very complex procedure to calculate the nonlinear transport of intense pulsed beams. Because the electrical potential of the beams depends on the beam dimensions, and the beam dimensions are related to the electric potential also, we can only solve the problem by iterations. Usually, we should provide the initial beam dimensions, and the first step: calculate the electric potential, next step: calculate particle trajectories, go to the first step… After several iterations we can obtain accuracy solutions.

References

1. Dragt A J. Lecture Notes on Nonlinear Orbit Dynamics, Summer School on High Energy Particle Accelerators, Fermi National Accelerator Laboratory, 1981, Carrigan R A et al., eds, New York, AIP87, 1982: 147-310

2. Goldstein H. Classical Mechanics, 2nd ed., Massachusetts, Addison -Wesley, 1980, 378-418.