MOTION OF CHARGED PARTICLE DENSE BUNCH IN NONUNIFORM EXTERNAL FIELDS

H. Ye. Barminova, ITEP, Moscow, Russia
A. S. Chikhachev, VEI, Moscow, Russia

Abstract

The kinetic distribution function was found allowed to describe the bunch dynamics in a selfconsistent manner ([1]-[3]) in the case of bunch formed as rotation ellipsoid with uniform charge density.

SPHERICAL BUNCH

Let us consider a bunch with spherical symmetry, coordinate system connecting with the bunch center. For the simplicity let consider that there are no external fields. In spherical coordinates r, θ, φ charged particle motion equations may be written:

$$mr^2\dot{\varphi}\sin^2\theta \equiv const = M_\varphi, \qquad (1)$$

$$m^2\left(r^2\dot{\theta}\right)^2 + \frac{M_\varphi^2}{\sin^2\theta} \equiv L > 0, \qquad (2)$$

$$\ddot{r} - a(t)r = \frac{L}{m^2 r^3}. \qquad (3)$$

Here M_φ, L are motion integrals followed from character of task symmetry, M_φ - projection of whole momentum on the axis z, $L-$ total momentum in the second power, m - particle mass. Equation (3) is correct only in case of sphere with uniform charge density, otherwise a depends not only on t, but on r. Using the relation $div\vec{E} = 4\pi en$, where e -charge, n -particle density, one can obtain for $a(t)$:

$$a(t) = e^2N / mR^3(t), \qquad (4)$$

here N is the total number of particles in the bunch, $R(t)$ - the bunch radius time-dependent.

For complete description of the bunch it is necessary to find three motion integrals. The 3rd integral may be found from (3), which describes the particle radial motion.

Let introduce

$$I = \left(R(t)\dot{r} - \dot{R}(t)r\right)^2 + \lambda\frac{R^2}{r^2} + \varepsilon_0^2\frac{r^2}{R^2}, \qquad (5)$$

$\left(\lambda = L/m^2\right)$. It is easy to obtain, that $dI/dt \equiv 0$, if

$$\ddot{R} - a(t)R = \frac{\varepsilon_0^2}{R^3(t)}. \qquad (6)$$

Distribution function for our bunch may be written as a function of motion integrals. Then calculating the particle density $n = \int \Phi dp_r dp_\theta dp_\varphi$ (where $p_r = m\dot{r}, p_\theta = mr\dot{\theta}, p_\varphi = mr\sin\theta\dot{\varphi}$), let replace integration variables to $I, \lambda = L/m^2, M_\varphi$. So for the density we obtain:

$$n = \frac{m^2}{4r^2R}\int_{\varepsilon_0^2\frac{r^2}{R^2}}^{\infty} dI \int_0^{\frac{r^2}{R^2}\left(I-\varepsilon_0^2\frac{r^2}{R^2}\right)} \frac{d\lambda}{\sqrt{I - \lambda R^2/r^2 - \varepsilon_0^2 r^2/R^2}} \times$$
$$\times \int_{-\sqrt{\lambda}\sin\theta}^{\sqrt{\lambda}\sin\theta} \frac{\Phi\left(I,\lambda,M_\varphi\right)dM_\varphi}{\sqrt{\lambda\sin^2\theta - M_\varphi^2/m^2}}. \qquad (7)$$

Limits of integration in (7) are determined in according with requirement for the expressions under integral signs to be more or equal to zero.

For density to be independent on θ distribution the function Φ must be independent on M_φ. So n looks as

$$n = \frac{\pi m^2}{4r^2R}\int_{\varepsilon_0^2\frac{r^2}{R^2}}^{\infty} dI \int_0^{\frac{r^2}{R^2}\left(I-\varepsilon_0^2\frac{r^2}{R^2}\right)} \frac{d\lambda\Phi(I,\lambda)}{\sqrt{\left(r^2/R^2\right)\left(I - \varepsilon_0^2 r^2/R^2\right) - \lambda}}. \qquad (8)$$

It should be noted that if we take Φ as $\Phi = \Phi_1(I)\Phi_2(\lambda)$, it leads to physically unreal result – negative density.

First found in [4], solution for Φ describing real physical conditions looks as:

$$\Phi(I,\lambda) = \frac{3N}{\pi^3 m\varepsilon_0^2}\frac{\sigma\left(1 - I/\varepsilon_0^2 + \lambda/\varepsilon_0^2\right)}{\sqrt{1 - I/\varepsilon_0^2 + \lambda/\varepsilon_0^2}}\sigma(\lambda). \qquad (9)$$

Here $\sigma(x)$ - Heaviside function. According to (8) and (9) one can obtain for density:

$$n = \frac{\pi m}{8rR^2}\frac{3N}{\pi^3 m\varepsilon_0^2}\int_0^{\varepsilon_0^2}\frac{d\lambda}{\sqrt{\lambda}} \times$$
$$\times \int_{\lambda\frac{R^2}{r^2}+\varepsilon_0^2\frac{r^2}{R^2}}^{\lambda+\varepsilon_0^2} \frac{dI}{\sqrt{I - \lambda\frac{R^2}{r^2} - \varepsilon_0^2\frac{r^2}{R^2}}\sqrt{\varepsilon_0^2 + \lambda - I}} \qquad (10)$$

In (10) upper limit of integration must be more than lower limit. Finally for density we obtain

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$$n=\frac{3N\pi^2 m}{8rR^2\pi^3\varepsilon_0^2 m}\frac{2\varepsilon r}{R}\sigma(R-r)= \frac{3N}{4\pi R^3(t)}\sigma\left(R(t)-r\right) \qquad (11)$$

ELLIPSOIDAL BUNCH

Spherical model is unsuitable for description of bunch behavior in external fields due to spherical symmetry absence. So the bunch model should be built in case of axial symmetry. The motion equations in cylindrical coordinates are

$$\ddot r+a_\perp(t)r=\frac{M_\varphi^2}{mr^3},\quad \ddot\varsigma+a_\parallel(t)\varsigma=0. \qquad (12)$$

Here $M_\varphi=rp_\varphi=mr^2\dot\varphi$ - motion equation integral, r - distance between particle position and axis, $\varsigma=z-z_0$ - projection of distance between particle and bunch center on axis z .

Equations (12) are correct for constant density bunch and permit to use the two motion integrals:

$$I_1=\left(R_\perp(t)\dot r-\dot R_\perp(t)r\right)^2+\mu^2\frac{R_\perp^2}{r^2}+\varepsilon_\perp^2\frac{r^2}{R_\perp^2},$$
$$I_2=\left(R_\parallel(t)\dot\varsigma-\dot R_\parallel(t)\varsigma\right)^2+\varepsilon_\parallel^2\frac{\varsigma^2}{R_\parallel^2}. \qquad (13)$$

Here $R_\perp(t)$, $R_\parallel(t)$ - transverse and longitudinal bunch sizes, $\varepsilon_\perp$, $\varepsilon_\parallel$ - transverse and longitudinal emittances, $\mu=\frac{M_\varphi}{m}$. Invariants I_1, I_2 are constant, if $R_\perp, R_\parallel$ satisfy to the equations:

$$\ddot R_\perp+a_\perp R_\perp=\varepsilon_\perp^2/R_\perp^3,\quad \ddot R_\parallel+a_\parallel R_\parallel=\varepsilon_\parallel^2/R_\parallel^3. \qquad (14)$$

The density $n=\int f dp_z dp_r dp_\varphi$ (where $p_z=m\dot z$, $p_r=m\dot r$, $p_\varphi=mr\dot\varphi=M_\varphi/r$) may be represented by means of motion integrals as follows:

$$n=\frac{m^3}{4}\int\Psi(I_1,I_2,\mu)\frac{d\mu}{r}\times \frac{dI_2}{R_\parallel\sqrt{I_2-\varepsilon_\parallel^2\frac{\varsigma^2}{R_\parallel^2}}}\frac{dI_1}{R_\perp\sqrt{I_1-\mu^2\frac{R_\perp^2}{r^2}-\varepsilon_\perp^2\frac{r^2}{R_\perp^2}}} \qquad (15)$$

Function $\Psi\geq 0$ must satisfy to relation $n=n_0\sigma\left(1-\frac{r^2}{R_\perp^2}-\frac{\varsigma^2}{R_\parallel^2}\right)$, i.e. Ψ must result in uniform density inside the bunch and must be equal to zero outside the bunch.

Let consider the next expression for Ψ :

$$\Psi(I_1,I_2,\mu)=\frac{3N}{2\pi^3 m^3}\frac{\varepsilon_\perp^2\left(1-I_2/\varepsilon_\parallel^2\right)^2}{\varepsilon_\parallel}\times \frac{\sigma\left(1-I_2/\varepsilon_\parallel^2\right)\sigma\left(\varepsilon_\perp^2\left(1-I_2/\varepsilon_\parallel^2\right)^2-\mu^2\right)}{\left(\varepsilon_\perp^2\left(1-I_2/\varepsilon_\parallel^2\right)^2-\mu^2\right)^{3/2}}\times \frac{\sigma\left(\varepsilon_\perp^2\left(1-I_2/\varepsilon_\parallel^2\right)+\mu^2/\left(1-I_2/\varepsilon_\parallel^2\right)-I_1\right)}{\sqrt{\varepsilon_\perp^2\left(1-I_2/\varepsilon_\parallel^2\right)+\mu^2/\left(1-I_2/\varepsilon_\parallel^2\right)-I_1}} \qquad (16)$$

Integrating with respect to I_1 and taking into account (16), finally for the density we obtain:

$$n=\frac{m^2}{4rR_\parallel R_\perp}\frac{3N}{2\pi^3 m^3\varepsilon_\parallel}\frac{2\pi r}{R_\perp}\int_{\varepsilon_\parallel^2\frac{\varsigma^2}{R_\parallel^2}}^{\varepsilon_\parallel^2}\frac{dI_2}{\sqrt{1-I_2/\varepsilon_\parallel^2-r^2/R_\perp^2}}\frac{1}{\sqrt{I_2-\varepsilon_\parallel^2\varsigma^2/R_\parallel^2}}= \frac{3N}{4\pi R_\perp^2 R_\parallel}\sigma\left(1-\frac{r^2}{R_\perp^2}-\frac{\varsigma^2}{R_\parallel^2}\right) \qquad (17)$$

Note that for a long time to build physically reasonable distribution function resulting in uniform density inside the ellipsoidal bunch was considered impossible ([3]). Such function was first built in [4].

SEMIAXIS EQUATIONS

Potential of the ellipsoid with uniform charge density in vacuum is performed by expression ([5]):

$$\Phi=-\frac{3eN}{4}\int_0^\infty\frac{dy}{T(y)}\left(1-\frac{z^2}{R_\parallel^2+y}-\frac{r^2}{R_\perp^2+y}\right),$$
$$T(y)=\left(R_\perp^2+y\right)\sqrt{R_\parallel^2+y} \qquad (18)$$

From (18) one can obtain relations for own fields. For the case $R_\parallel>R_\perp$ we can obtain:

$$E_r=\frac{3eNr}{2\left(R_\parallel^2-R_\perp^2\right)^{3/2}}\left(\frac{R_\parallel\sqrt{R_\parallel^2-R_\perp^2}}{R_\perp^2}-\frac{1}{2}\ln\frac{R_\parallel+\sqrt{R_\parallel^2-R_\perp^2}}{R_\parallel-\sqrt{R_\parallel^2-R_\perp^2}}\right)= -\frac{ma_\perp(t)r}{e} \qquad (19)$$

$$E_z=\frac{3eN\varsigma}{2\left(R_\parallel^2-R_\perp^2\right)^{3/2}}\left(\ln\frac{R_\parallel+\sqrt{R_\parallel^2-R_\perp^2}}{R_\parallel-\sqrt{R_\parallel^2-R_\perp^2}}-2\frac{\sqrt{R_\parallel^2-R_\perp^2}}{R_\parallel}\right)= -\frac{ma_\parallel(t)\varsigma}{e} \qquad (20)$$

$a_\perp(t)$ and $a_\parallel(t)$ take part in the motion equations and in the equations for the semiaxes $R_\perp$, $R_\parallel$ (see (14)).

Further, let suppose there is an external field in the region too, such as $div\vec{E}^{ext}=0$. In the paraxial approximation external field potential may be performed by the next way: $\Phi^{ext}\cong\Phi_0(z)-\frac{r^2}{4}\Phi_0''(z)$. Let use paraxial approximation for the external field force too. So in (14) $a_\parallel(t)$ must be replaced by the expression $a_\parallel(t)+\frac{e}{m}\Phi_0''(z(t))$, and $a_\perp(t)$ respectively by $a_\perp(t)-\frac{e}{2m}\Phi_0''(z(t))$. If there is the external field growth, transverse direction focusing appears, and on the contrary, the phase variation increases, e. g. longitudinal defocusing.

The generalization to the case of the bunch motion in weakly nonuniform magnetic field is relatively simple. In the left part of equation for $R_\perp$ the next term must be added: $R_\perp^2\left(eB(t)/mc\right)^2$. Here $B(t)=B(z(t))$ is external stationary magnetic field. So equation for $R_\parallel$ is not changed, and the longitudinal magnetic field changes the longitudinal bunch size implicitly by changing of the transverse size.

ZERO LONGITUDINAL EMITTANCE

Eq. (16) does not allow to describe the bunch with zero longitudinal emittance due to $\varepsilon_\parallel$ placed in denominator. So in this section another, rather simple method of distribution function construction will be considered, resulting in uniform charge density of ellipsoidal bunch.

In case of $\varepsilon_\parallel=0$ (14) is followed by conclusion that I_2 is a linear invariant in the second power. If $\ddot{\varsigma}-a_\parallel(t)\varsigma=0$, then $I=S(t)\dot{\varsigma}-\dot{S}(t)\varsigma\equiv const$, when S satisfies to $\ddot{S}-a_\perp(t)S=0$. The last has two linearly independent solutions, we sign them $R_\parallel$ and S_1. Let consider their time-independent Wronskian equal to 1: $S(t)\dot{R}_\parallel(t)-\dot{S}(t)R_\parallel(t)\equiv 1$.

Let denote $I_2^{(1)}=S_1\dot{\varsigma}-\dot{S}_1\varsigma,\quad I_2^{(2)}=R_\parallel\dot{\varsigma}-\dot{R}_\parallel\varsigma.$

Distribution function will be represented in the form of two δ - functions product:

$$\Psi\left(I_1,I_2^{(1)},I_2^{(2)}\right)=const\,\delta\left(I_1+\left(I_2^{(1)}\right)^2-1\right)\delta\left(I_2^{(2)}\right),\quad(21)$$

I_1 is determined in (13). After integration with respect to the $\dot{\varsigma}$ we will obtain one δ - function with argument

$$I_1+\varsigma^2\left(\frac{\dot{R}_\parallel S-R_\parallel\dot{S}}{R_\parallel}\right)^2-1=\left(\dot{r}R_\perp-r\dot{R}_\perp\right)^2+\mu^2\left(\frac{R_\perp}{r}\right)^2+\varepsilon_\perp^2+\frac{\varsigma^2}{R_\parallel^2}-1$$

After integration with respect to the $\dot{r}$ and μ we obtain for the density $n=const\frac{\pi}{R_\perp^2R_\parallel}\sigma\left(1-\frac{r^2}{R_\perp^2}-\frac{\varsigma^2}{R_\parallel^2}\right)$, which yields the following expression: $const=\frac{4N}{3\pi^2}$.

Note that the model constructed by the same method can describe the bunch without axial symmetry[6].

Combined usage of the linear and square invariants for 3D-bunches was considered in [7].

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