COVARIANT FORMULATION OF THE VLASOV EQUATION

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Abstract

In traditional approach, the Vlasov equation is considered as integro-differential equation with nonlinear term accounting for the electromagnetic interaction. That formulation includes partial derivatives on phase coordinates.

According to the covariant approach, physical relations should be presented by tensor equations. The main feature of the covariance is that any tensor equation can be written without using of coordinates.

In covariant formulation of the Vlasov equation, we use such tensor objects as Lie derivatives. Classical and relativistic cases are described similarly. A difference between these two cases appears only in form of particle motion equations.

Another feature of presented approach is consideration of degenerate distributions in the phase space. By degenerate distribution we mean a distribution which have support of dimension smaller than dimension of the phase space. The simplest case of degenerate distribution is the distribution described by the Dirac measure. Another example is the Kapchinsky-Vladimirsky distribution, when particles are distributed on the 3-dimensional surface in the 4-dimensional phase space.

Presented results can be applied for description and simulation of charge particle distributions for high-intensity beam.

PHASE SPACE

Consider a domain D in 4-dimensional spacetime and an observer which can measure time of each event in D : $t(x)$, $x \in D$. Then D can be represented as union of disjoint layers corresponding to various values of t. Let's call such layers the layers, or spaces, of simultaneous events. In classical theory, each such layer is the same for different observers. Assume that $t(x)$ is continuous mapping and the spaces of simultaneous events can be described as surfaces of class C_1 in some system of coordinates with slowly varying coordinates [1].

Let us call an observer and a system of mapping of corresponding layers of simultaneous events to some selected layer the reference frame. The selelected layer of simultaneous space is called the configuration space associated with the reference frame [1].

When time passes, particles move from one layer of simultaneous events to another, but we can examine dynamics of particle ensemble in 3-dimensional configuration space of some frame of reference.

Let us consider the 6-dimensional tangent bundle of the configuration space as the phase space. Denote by x a position in the configuration space, and by q a position in the phase space. Assume that coordinates in the phase space can be chosen so that the first 3 coordinates coincide with the coordinates in the configuration space, and that the rest 3 coordinates allow us to find the velocity vector in the configuration space:

$$\frac{dx}{dt} = V(x^1, x^2, x^3, q^4, q^5, q^6).$$

Though we use the reference frame, the proposed approach is fully covariant as we can consider instead of system of layers of simultaneous events every system of smooth timelike surfaces filling the domain D, introduce their continuous parametrization and system of their mapping to some selected surface, and consider particle dynamics in the tangent bundle of that surface.

PARTICLE DISTRIBUTION DENSITY

We shall consider various types of distributions, which will be described on the basis of the common approach.

In the simplest case, assume that we deal with continuos charged media occupying a domain G_0 in the phase space instead of set of discrete particles. Consider a family of subdomains $\{G\}$, $G \subset G_0$, with smooth boundaries for which their characteristic functions are defined:

$$\chi_G(q) = \begin{cases} 1, & q \in G \\ 0, & q \notin G. \end{cases}$$

Let us call differential form of 6-th degree

$$n = n_{123456}(t,q)dx^1 \wedge dx^2 \wedge dx^3 \wedge dq^4 \wedge dq^5 \wedge dq^6, \quad (1)$$

the particle density distribution in the phase space (or phase density) if for each subdomain G

$$\int\limits_{G_0} \chi_G(q) n(q) = N_G. \quad (2)$$

Here N_G is the number of particles in G, which in this model may be not integer. For simplicity, assume that $n = n_{123456}(t,q)$ is continuously differentiable. Cases of piece-wise continuous and piece-wise differentiable component can be considered analogously.

Consider the space of functions $f(q)$ for which $\int_G f(q)\omega(q)$ exists for any form of 6-th degree $\omega(q)$ from given class. Let us call such functions integrable and denote by $\mathcal{F}$ their space. For some form $\omega(q)$, define a linear functional on $\mathcal{F}$ by the rule

$$< \omega, f >= \int\limits_{G_0} f(q)\omega(q), \quad f \in \mathcal{F}. \quad (3)$$

Then definition (2) can be written as

$$< n, \chi >= N_G. \quad (4)$$

Let us consider now the discrete model of point-like particles. In the frames of this model each particle is represented by a point in the phase space. Let us introduce the linear functional $\delta(q)$ on $\mathcal{F}$:

$$< \delta(q), f >= f(q), \quad f \in \mathcal{F}. \quad (5)$$

As the measure $\mu_D =< \delta(q), \chi_D >$ is called the Dirac measure, let us call the functional (5) the density of the Dirac measure. The space of such functionals is linear, their linear combination being

$$< \sum_i \alpha_i \delta(q_{(i)}), f >= \sum_i \alpha_i f(q_{(i)}).$$

For the model of point-like particles, let us call a linear combination of functionals (5), such that for each subdomain G the equality (4) holds, the phase density. It is easy to see that in this case $\alpha_i = 1$, and $q_{(i)}$ are particle positions in the phase space, $i = \overline{1, N}$ where N is the total number of particles:

$$n(q) = \sum_{i=1}^{N} \delta(q_{(i)}). \quad (6)$$

In this case, the density (6) is described by a scalar function, which is a differential form of 0 degree.

Consider also the model that can be regarded as intermediate case between the model on continuous media and the model of point-like particles. Assume that particles are continuously distributed on some oriented surface S in the domain G_0. The Kapchinsky-Vladimirsky distribution and the Brillouin flow are examples of such distributions. We shall describe distribution density in this case by a differential form of m-degree defined on the surface. This form depends on orientation of the surface, which is given by a set of $n - m$ vectors: a change of the orientation can result in change of the form component sign. Assume for simplicity that form components are continuously differentiable functions of coordinates on the surface.

A form of m-degree $\sigma(q)$ defined on a m-dimensional oriented surface set a functional on $\mathcal{F}$:

$$< \sigma(q), f >= \int_S f(q)\sigma(q).$$

In this case, call such form

$$n(q) = \sigma(q), \quad (7)$$

that the condition (4) holds, the phase density.

THE VLASOV EQUATION

According to Vlasov, assume that particle dynamics is determined by an external electromagnetic field and by the self electromagnetic field, which is created by the media being used as the model of a particle ensemble. For continuous models (1), (7), we assume that particle density has sufficiently small components to neglect the collision integral.

The particle dynamics equations define vector field w in the domain D_0 of the phase space. If right hand sides of the dynamics equations are continuously differentiable, then there exist integral lines, unique for each point and each instance of time. Time can be taken as a parameter for integral lines. In the simple case, when the phase space is associated with an inertial frame, the dynamics equations take the form

$$\frac{dx}{dt} = v, \quad (8)$$

$$m \sum_{i=1}^{3} g_{ik} (\frac{d}{dt}\gamma v)^i = eE_k + e \sum_{i=1}^{3} B_{ki} v^i, \; k = 1, 2, 3. \quad (9)$$

Here e and m are charge and mass of a particle, γ is reduced energy (in nonrelativistic case $\gamma = 1$), g_{ik} are components of the metric tensor. In nonrelativistic case, metric tensor is defined in the configuration space. In relativistic case, $g_{00} = 1$, $g_{0i} = 0$, $i = 1, 2, 3$, so that components of covariant derivation of the velocity vector in left hand side of (9) contains Christoffel symbols only with spatial indices. E is the first degree form of the electric field intensity and B is the second degree form of the magnetic flux density.

Let us consider Lie dragging $F_{w,\,\delta\lambda}$ along vector field w, which maps each point q to point shifted along integral line by parameter increment δt. It induces a coordinate transform, which can be considered as shift of coordinate system: for every point q we take as its coordinates coordinates of its preimage at Lie dragging [2]. Then Lie dragging $F_{w,\,\delta t}T$ of tensor T can be defined as follows: components of $F_{w,\,\delta t}T$ in shifted coordinates are equal to corresponding components of tensor T in initial coordinates. Lie derivative of tensor field T along vector field w can be defined as

$$\mathcal{L}_w T = \lim_{\delta t \to 0} \frac{T - F_{w,\,\delta t}T}{\delta t}, \quad (10)$$

if the limit exists.

For differential form of p degree, which is covariant tensor, the components of Lie derivatives are equal

$$(\mathcal{L}_w T)_{i_1 \ldots i_p} = \frac{\partial T_{i_1 \ldots i_p}}{\partial q^k} w^k + \frac{\partial w^j}{\partial q^{i_1}} \cdot T_{j\, i_2 \ldots i_p} + \ldots$$

$$\ldots + \frac{\partial w^j}{\partial q^{i_p}} \cdot T_{i_1 \ldots i_{p-1}\, j}, \quad (11)$$

Consider the equation for density distribution form. It is easy to see that dimension of the distribution supporter does not change, because dragged basis vectors can be use as basis vector in dragged point. Assume that particles don't arise or destroy. Then integrating on each domain of the phase space or surface where particles are located gives the same result as integration on dragged domain or surface. It means, in accordance with definition of Lie dragging, that evolution of particle density form can be described as Lie dragging of the density form along vector field w, which is defined by the dynamics equations:

$$n(t+\delta t, F_{w,\delta t}q) = F_{w,\delta t}n(t,q). \qquad (12)$$

We shall call this equation covariant form of the Vlasov equation.

How does density form change in the given point of the phase space? In the case when distribution is described by the form of 6 degree with continuously differentiable components or when degree of the density form less then 6, but the surface where particles are located does not change, one can write equation with partial derivatives for the density form. Let phase density form is equal to $n(t,q)$ at some instance t in a point q. Then it will be equal to $n(t+\delta t, q) = F_{w,\delta t}n(t, F_{w,-\delta t}q)$ at instance $t+\delta t$, as it changes according to the equation (12). Introducing the form derivative on a parameter as form which component are derivatives of corresponding components on this parameter, we obtain the Vlasov equation in the form

$$\frac{\partial n}{\partial t} = \lim_{\delta t\to 0}\frac{n(t+\delta t,q)-n(t,q)}{\delta t} = -\mathcal{L}_w n(t,q). \qquad (13)$$

As a simple example, consider an ensemble of non-relativistic particles, which dynamics is described by the equations (8), (9), $\gamma = 1$, and the particles distribution is described by the form of 6 degree. Take spatial Cartesian coordinates and corresponding components of velocity $q^{i+3} = v^i$, $i = 1,2,3$ as coordinates in phase space.

According to (9), force components don't depend on corresponding components of velocity. Then in right hand side of (11) we should take only first term, and the Vlasov equation takes the form

$$\frac{\partial\tilde{n}}{\partial t} + \sum_{i=1}^{3} v^i\frac{\partial\tilde{n}}{\partial x^i} + \sum_{i=1}^{3}\frac{e}{m}(E_i + e\sum_{i=1}^{3}B_{ij}v^j)\frac{\partial\tilde{n}}{\partial v^i} = 0.$$

Here $\tilde{n}$ denotes corresponding component of the phase density n.

Calculation of the self electromagnetic field can be carried out by various ways. For example, for distribution (6) field can be calculated as superposition of the fields of moving charged particles.

In general, self electromagnetic field can be found as a solution for the Maxwell equations, which contain in non-relativistic case the charge density and the current density forms and in relativistic case the current density form of 3 degree.

For distributions (7) description of the charge density and the current density can require introducing functionals analogous to functionals introduced for description of the phase density. Nevertheless, for some cases, e.g. KV distribution, we can do without new functionals.

Assume that current density form has smooth components. Then we can construct mapping which transforms the phase density form to the current density form. For example, when the phase density is described by the expression (1), it can be shown that components of the current density are

$$J_{x^1,x^2,x^3}(t,x) = \int\limits_{D(x)} n_{123456}(t,q)dq^4\wedge dq^5\wedge dq^6,$$

$$J_{t,x^1,x^2}(t,x) = \int\limits_{D(x)} n_{123456}(t,q)\frac{dx^3}{dt}dq^4\wedge dq^5\wedge dq^6,$$

$$J_{t,x^1,x^3}(t,x) = \int\limits_{D(x)} n_{123456}(t,q)\frac{dx^2}{dt}dq^4\wedge dq^5\wedge dq^6,$$

$$J_{t,x^2,x^3}(t,x) = \int\limits_{D(x)} n_{123456}(t,q)\frac{dx^1}{dt}dq^4\wedge dq^5\wedge dq^6.$$

Here $D(x)$ denotes the set of admissible values of the phase coordinates q^4, q^5, q^6 in the point x of the configuration space.

CONCLUSION

Covariant form of the Vlasov equation (12) is presented. All kinds of distributions are described by differential forms of various degrees. For all of them, distribution density is defined by the equality (4) and satisfies the covariant equation (12). The equation (12) contains tensor objects — distribution density form and Lie dragging, t being a parameter. Instead of time t we can take any parameter for the set of timelike surfaces filling some domain in the space-time.

Such approach can be applied also to the Liouville equation, which can be consider as the partial case when interaction is neglected.

REFERENCES

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