Tutorial: Introduction to Free-Electron Laser Theory

Agostino Marinelli[1]

and Claudio Pellegrini[1,2]

[1]SLAC National Accelerator Laboratory

[2]Department of Physics and Astronomy, UCLA

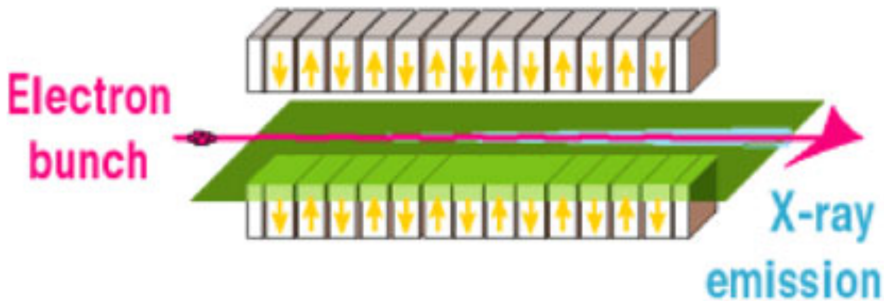
Outline

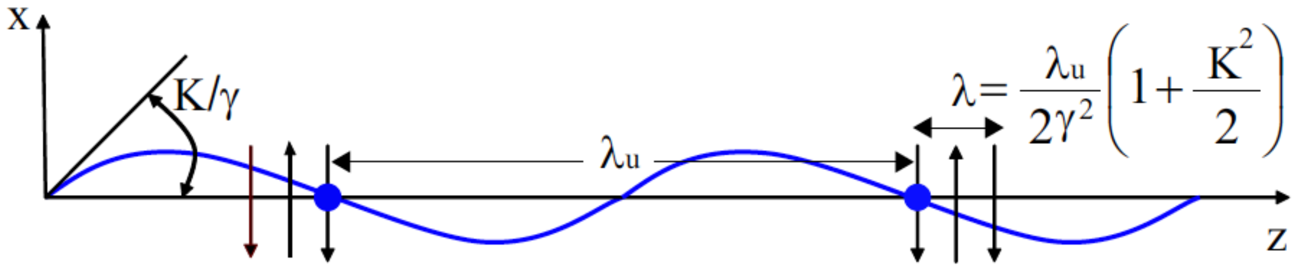
-Basic princples

-1-D theory

-Introduction to 3-D theory

Undulator radiation, single electron





$$\lambda = \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{K^2}{2}\right)$$

$$K = \frac{eB_0\lambda_U}{2\pi mc^2}$$

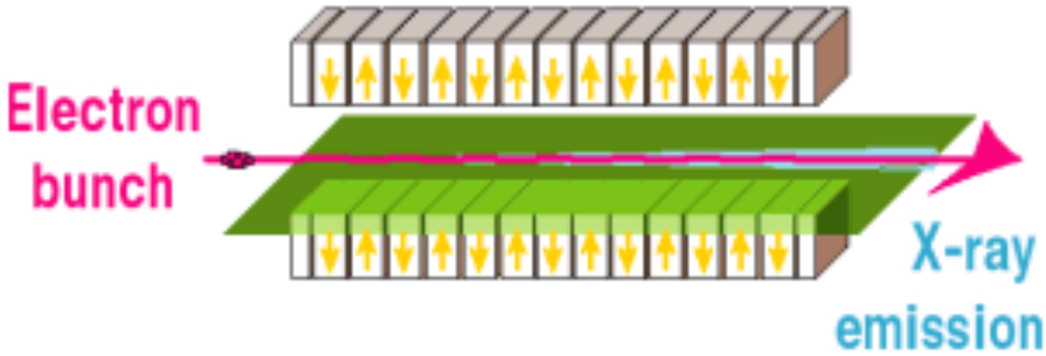
On Axis

$$\lambda \simeq \frac{\lambda_U(1 + K^2/2)}{2\gamma^2}$$

On resonance:

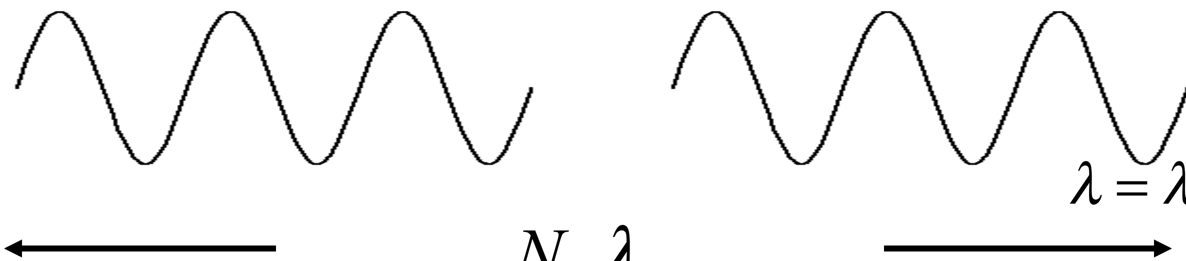
Radiation slips ahead by one wavelength per undulator period

Undulator radiation, single electron



Undulator with N_U periods.

$$\Delta\lambda / \lambda \simeq 1 / N_U$$



$$\lambda = \lambda_U (1 + K^2 / 2 + \gamma^2\theta^2) / 2\gamma^2$$

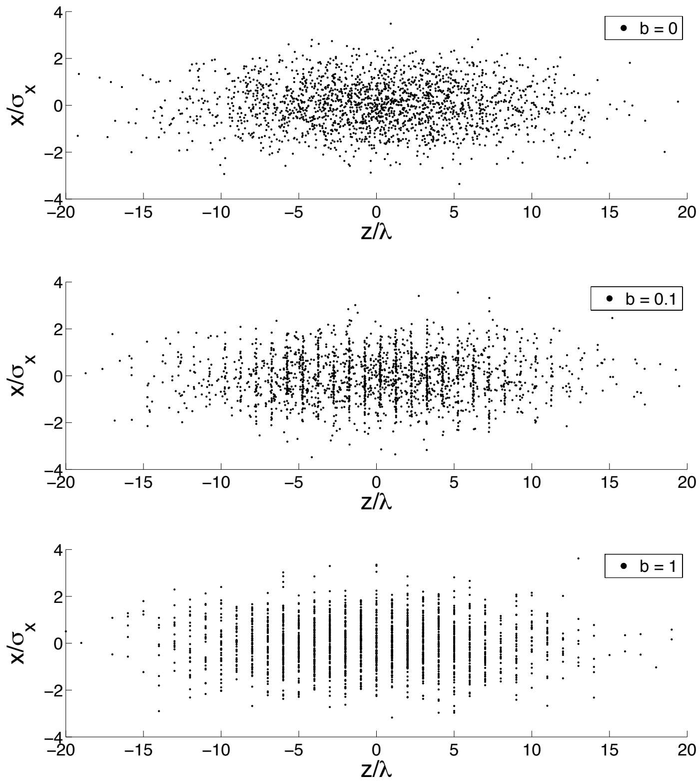
$$N_U \lambda$$

Each electron emits a wave train with N_U cycles

For a case like that of LCLS: $\gamma \approx 3\times10^4,\ \lambda_U \approx 3cm,\ K \approx 3,\ N_U \approx 3500$

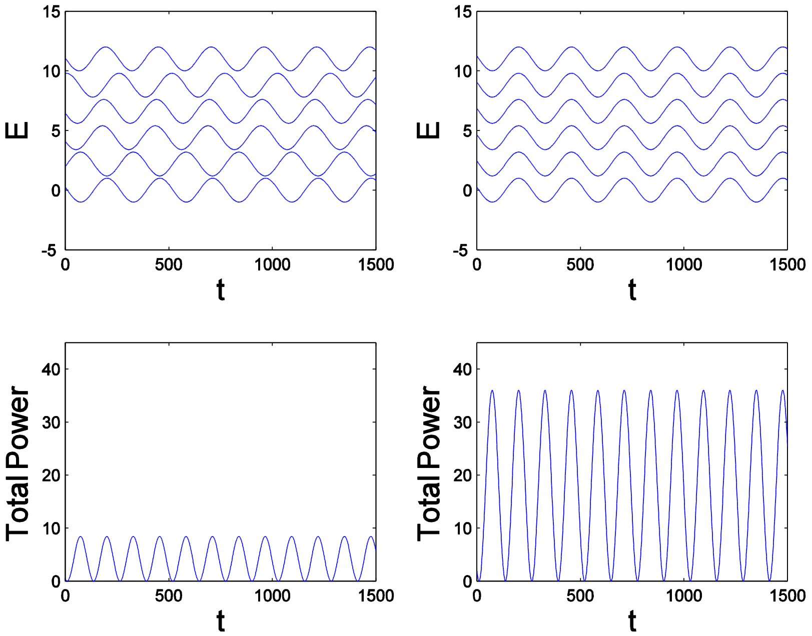
$\lambda = 0.1nm,\ \Delta\omega/\omega \approx 3\times10^{-4},\ \lambda N_U = 0.3\mu m(1fs)$

FEL: Working Principle



$$\frac{dP}{d\Omega d\omega} = \frac{dP_{sp}}{d\Omega d\omega} F(\vec{k}_\perp) N^2 \, |b(k)|^2$$

$$b(k) = \frac{1}{N} \sum_n e^{-ikz_n}$$



Working Principle

Resonant Interaction

↓

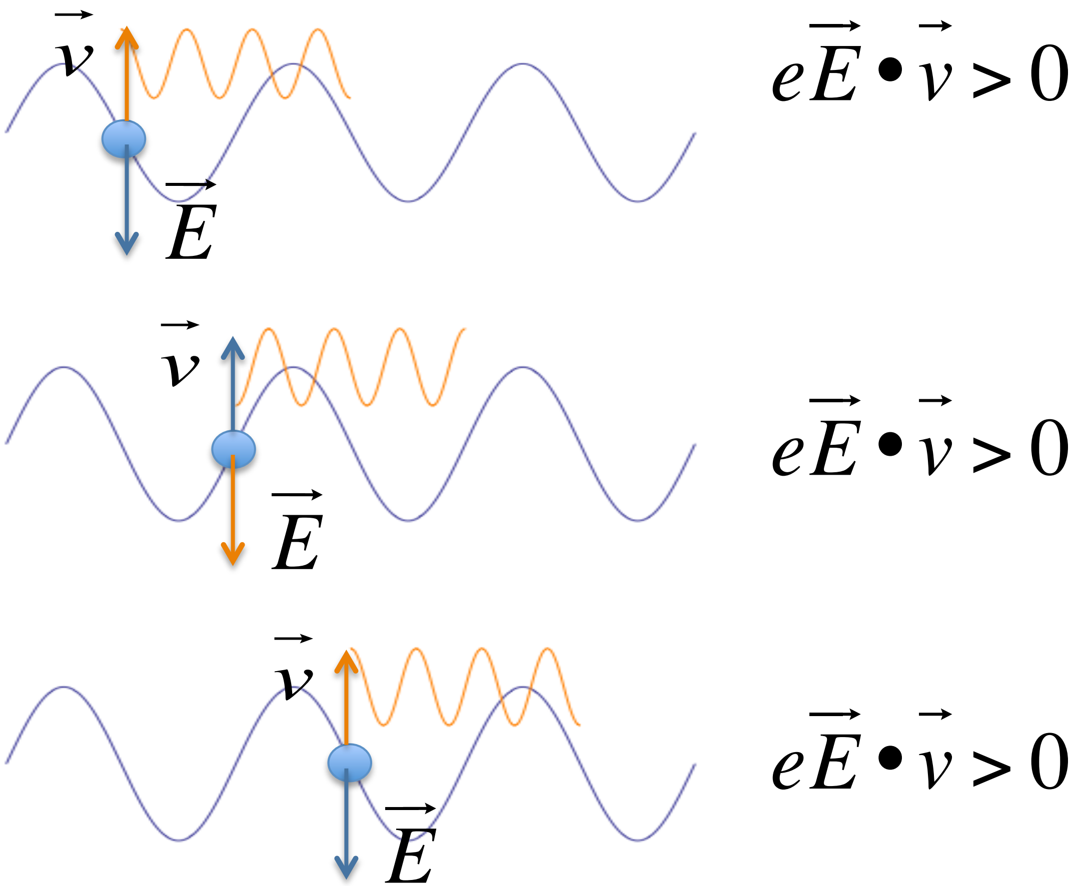
Energy Modulation

↓

Density Modulation

↓

Coherent Radiation



$$\lambda_r = \lambda_w \frac{1 + K^2}{2\gamma^2}$$

Light slips ahead by 1 wavelength per oscillation period

Working Principle

Resonant Interaction

↓

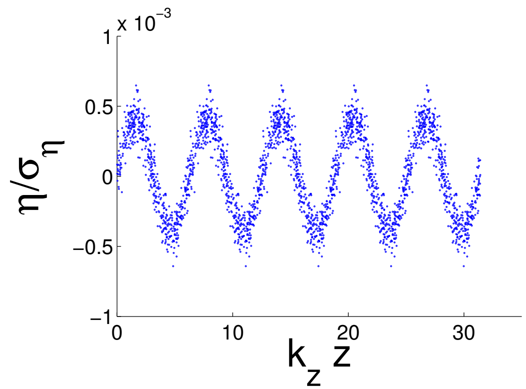
Energy Modulation

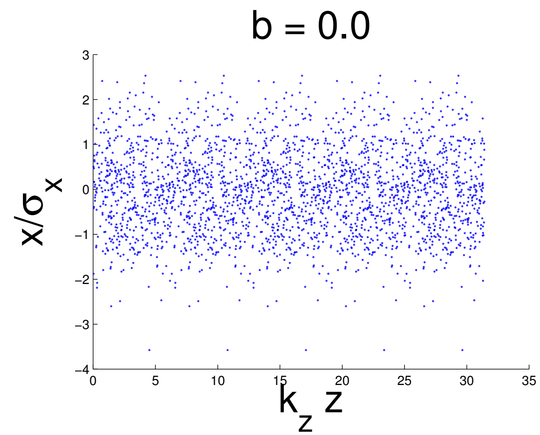
↓

Density Modulation

↓

Coherent Radiation





Application to x-rays severely limited by mirror availability

Working Principle

Resonant Interaction

↓

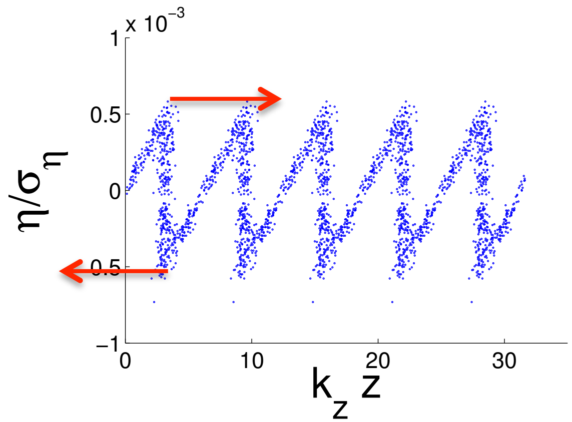
Energy Modulation

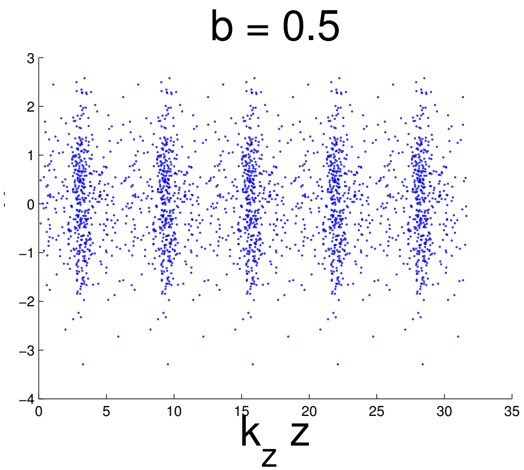
↓

Density Modulation

↓

Coherent Radiation





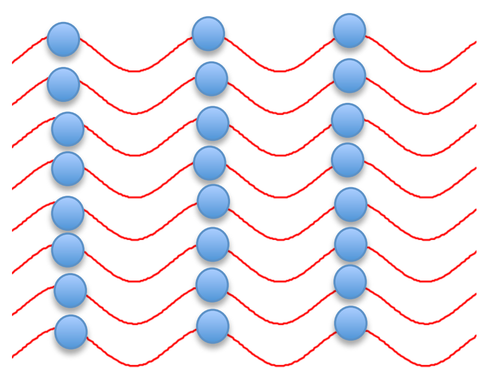
Working Principle

Resonant Interaction

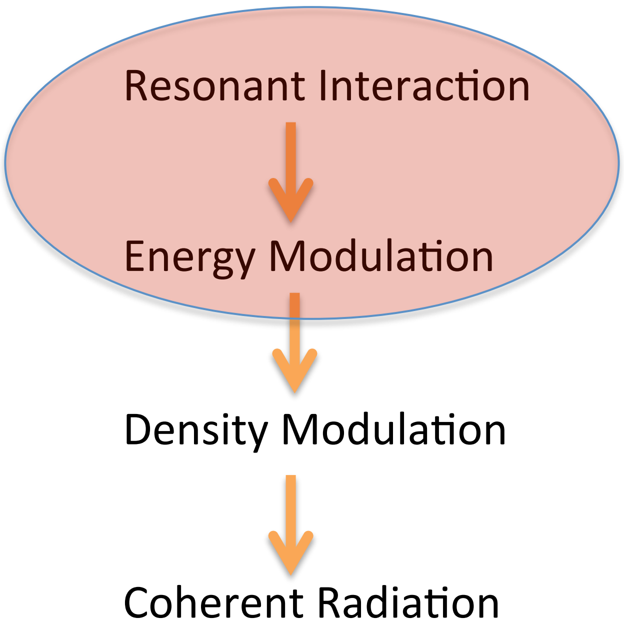
Energy Modulation

Density Modulation

Coherent Radiation



Linear FEL Equations

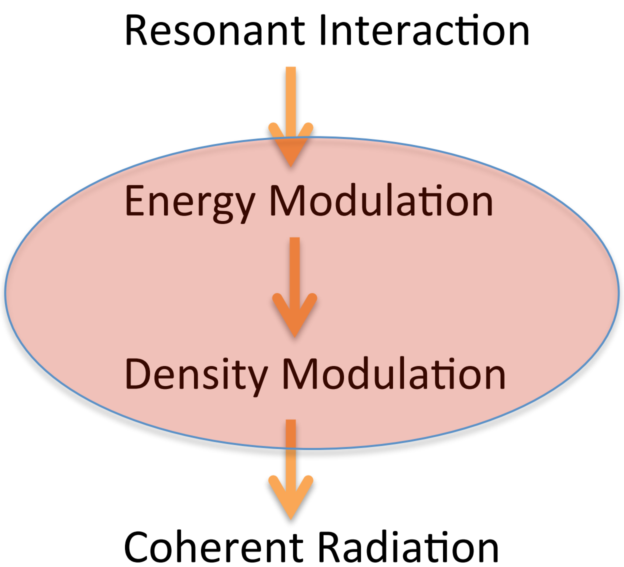


$$\frac{d}{dz}\tilde{\eta} = \frac{eK\bar{E}}{2mc^2\gamma_b^2}$$

$$\eta = \frac{\gamma - \gamma_b}{\gamma_b} \ll 1$$

$$\tilde{\eta} = \frac{1}{N}\sum_{n=1}^{N} \eta_n \exp\left(-i\theta_{n,0}(1+\Delta)\right)$$

Linear FEL Equations



$$\frac{d}{dz} b = -2ik_w \tilde{\eta}$$

Linear FEL Equations

Resonant Interaction

↓

Energy Modulation

↓

Density Modulation

↓

Coherent Radiation

$$(k - k_r)/k$$

$$\left(\frac{d}{dz} + ik_w\Delta\right)\bar{E} = -\frac{eKn_0}{2\gamma_b\epsilon_0}(b)$$

Assumptions

- Neglect diffraction
- Small signal ($b \ll 1$)
- Slowly varying envelope (i.e. narrow bandwidth signal)
- No velocity spread (longitudinal and transverse)

Exponential Growth @ Resonance

$$\Delta = 0$$

$$\bar{E}, b, \tilde{\eta} \propto \exp(-2ik_w \alpha z)$$

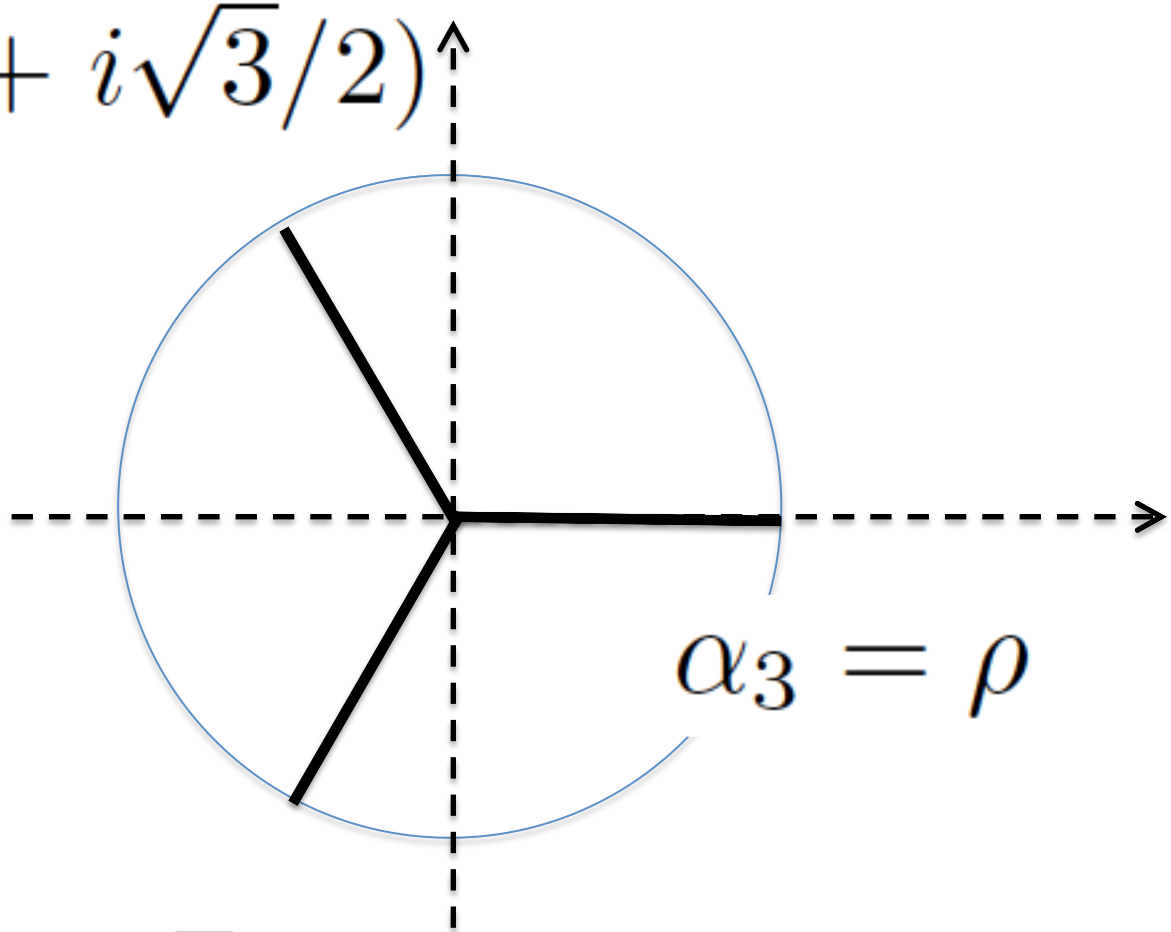
$$\downarrow$$

$$\alpha^3 = \rho^3$$

$$\rho = (Kk_p/4k_w)^{(2/3)}$$

Roots

$$\rho(-1/2 + i\sqrt{3}/2)$$

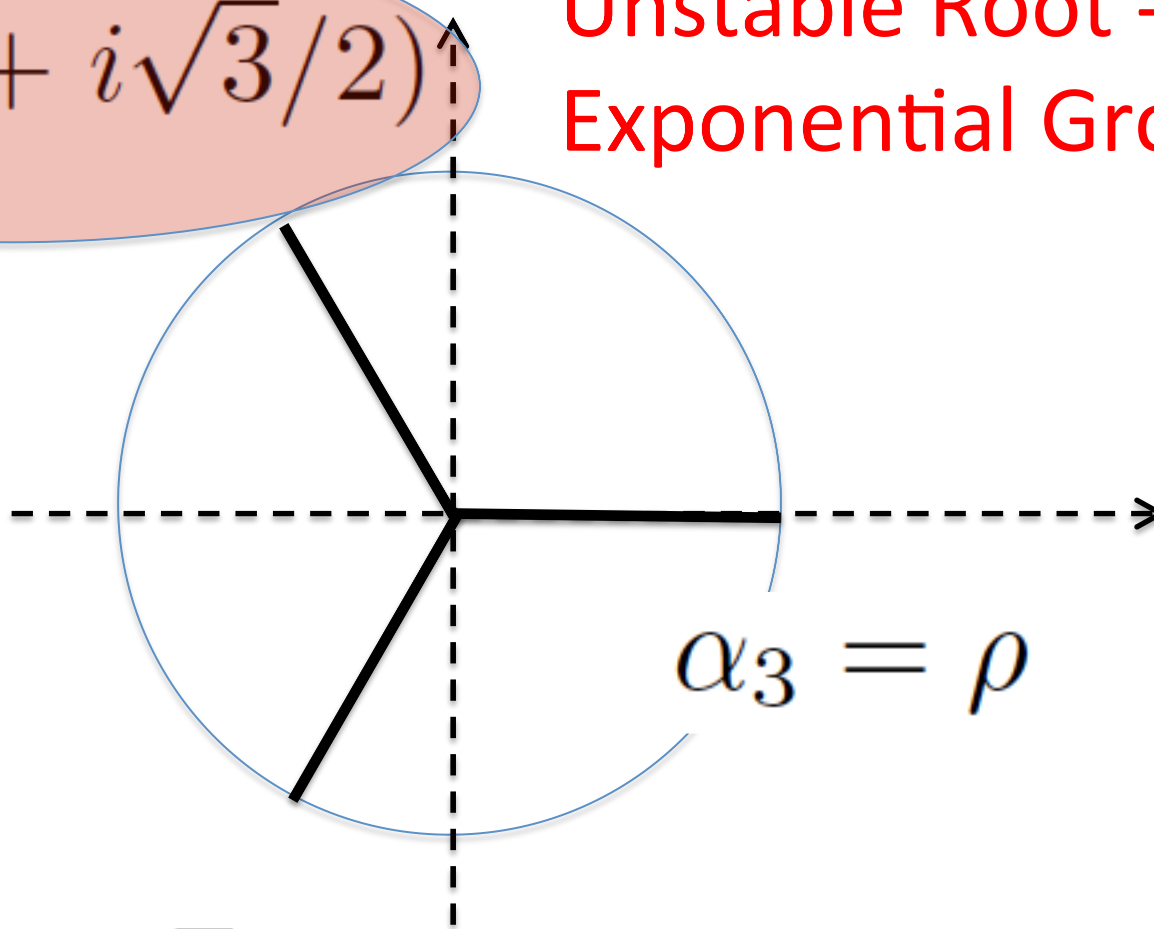


$$\alpha_3 = \rho$$

$$-\rho(1/2 + i\sqrt{3}/2)$$

Roots

Unstable Root -> Exponential Growth



The ρ parameter

$$\rho = (Kk_p/4k_w)^{(2/3)}$$

$\propto n_e^{1/3}$ High density -> higher gain! (note: scaling typical of all 3-wave instabilities…)

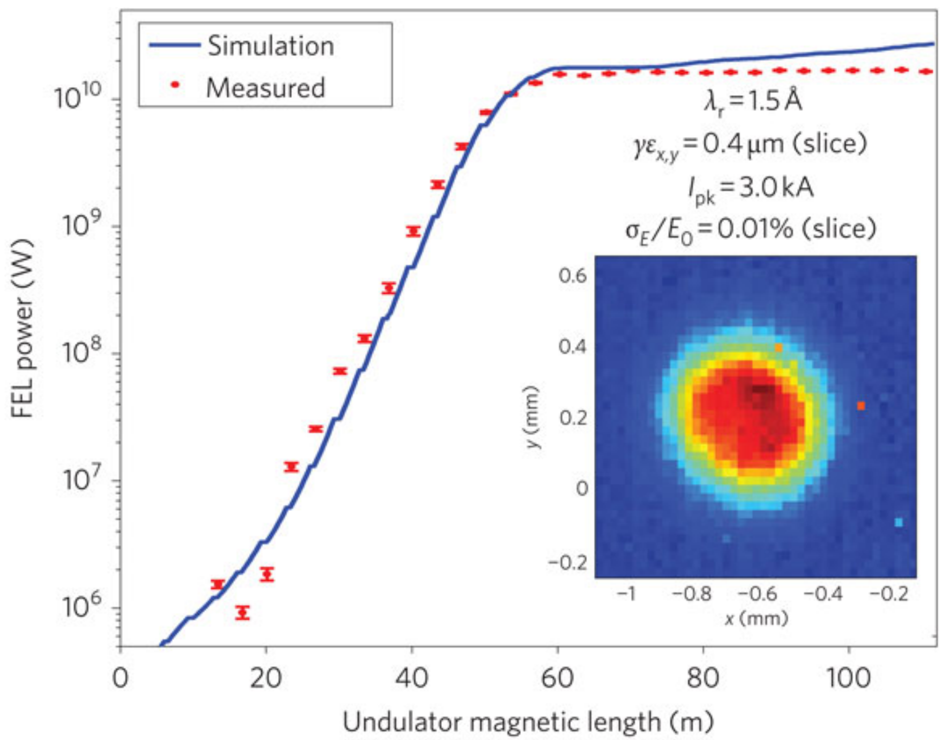
$\propto 1/\gamma$ Smaller growth rate at higher energies

$\propto K^{2/3}$ Stronger magnetic field -> higher gain

Typically 10^{-3} to 10^{-4} for x-ray parameters

That's Pretty Much it…

SLAC

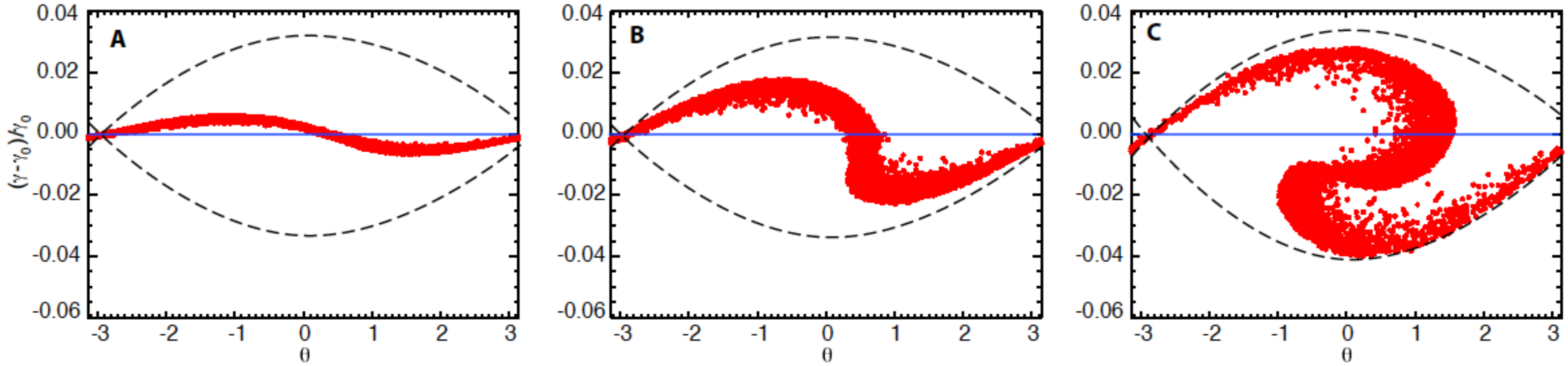


nature photonics ARTICLES
PUBLISHED ONLINE: 1 AUGUST 2010 | DOI: 10.1038/NPHOTON.2010.176

First lasing and operation of an ångstrom-wavelength free-electron laser

P. Emma[1]*, R. Akre[1], J. Arthur[1], R. Bionta[2], C. Bostedt[1], J. Bozek[1], A. Brachmann[1], P. Bucksbaum[1], R. Coffee[1], F.-J. Decker[1], Y. Ding[1], D. Dowell[1], S. Edstrom[1], A. Fisher[1], J. Frisch[1], S. Gilevich[1], J. Hastings[1], G. Hays[1], Ph. Hering[1], Z. Huang[1], R. Iverson[1], H. Loos[1], M. Messerschmidt[1], A. Miahnahri[1], S. Moeller[1], H.-D. Nuhn[1], G. Pile[3], D. Ratner[1], J. Rzepiela[1], D. Schultz[1], T. Smith[1], P. Stefan[1], H. Tompkins[1], J. Turner[1], J. Welch[1], W. White[1], J. Wu[1], G. Yocky[1] and J. Galayda[1]

What Happens at Saturation?



$$\frac{d}{dz} b = -2ik_w \tilde{\eta}$$

$$|\tilde{\eta}| = \rho |b|$$

@ saturation b ~ 1 → $|\tilde{\eta}_{sat}| \sim \rho$

What Happens at Saturation?

$$P_{rad} = Z_0|\bar{E}|^2 = \rho P_b|b|^2$$

@ saturation b ~ 1

↓

$$P_{sat} \sim \rho P_b$$

For typical HXR FELs ~ 10-100 GW

Normalized FEL Equations

SLAC

Normalize everything to saturation value

$$\frac{d}{d\bar{z}}a + i\frac{\Delta}{2\rho}a = -b$$

$$\frac{d}{d\bar{z}}b = -ip$$

$$\frac{d}{d\bar{z}}p = a$$

$$p = \frac{\tilde{\eta}}{\rho}$$

$$a = \bar{E}\sqrt{\frac{Z_0}{P_b}}$$

$$\bar{z} = 2k_w\rho z$$

Normalized FEL Equations

SLAC

Natural scaling of detuning is also ~ ρ

Normalize everything to saturation value

$$\frac{d}{d\bar{z}}a + i\frac{\Delta}{2\rho}a = -b$$

$$\frac{d}{d\bar{z}}b = -ip$$

$$\frac{d}{d\bar{z}}p = a$$

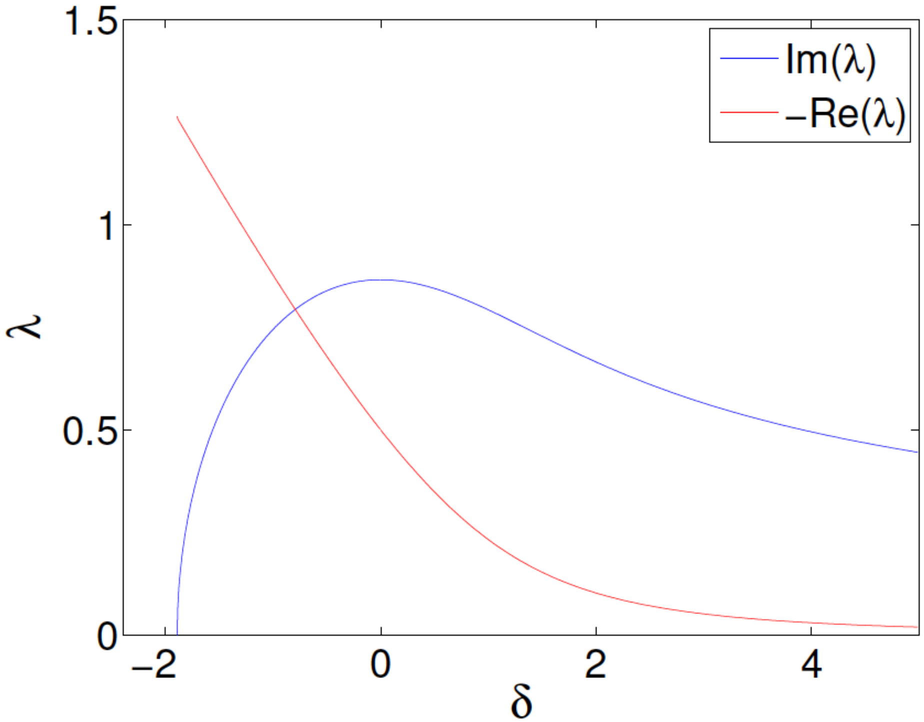
$$p = \frac{\tilde{\eta}}{\rho}$$

$$a = \bar{E}\sqrt{\frac{Z_0}{P_b}}$$

$$\bar{z} = 2k_w\rho z$$

Dispersion Relation for General Detuning

$$\lambda^3 - \delta\lambda^2 - 1 = 0$$



Assume

$$\sim \exp(-i\lambda\bar{z})$$

And a finite detuning

$$\delta = \frac{k - k_r}{2k_r\rho}$$

To 2nd Order…

$$\lambda_1 = -\frac{1}{2} + \frac{\delta}{3} - \frac{\delta^2}{18} + i\frac{\sqrt{3}}{2}\left(1 - \frac{\delta^2}{9}\right)$$

$$\frac{d}{d\delta}\lambda = \frac{\lambda}{3\lambda - 2\delta}$$

$$\frac{d^2}{d\delta^2}\lambda = \frac{\frac{d}{d\delta}\lambda}{3\lambda - 2\delta} - \frac{\lambda(3\frac{d}{d\delta}\lambda - 2)}{(3\lambda - 2\delta)^2}$$

To 2nd Order...

SLAC

$$\lambda_1 = -\frac{1}{2} + \frac{\delta}{3} - \frac{\delta^2}{18} + i\frac{\sqrt{3}}{2}\left(1 - \frac{\delta^2}{9}\right)$$

$$\sigma_\omega/\omega = 6\rho/\sqrt{(2\sqrt{3}\bar{z})}$$

Bandwidth ~ z ½

@ saturation

Δω/ω ~ ρ

To 2nd Order...

$$\lambda_1 = -\frac{1}{2} + \frac{\delta}{3} - \frac{\delta^2}{18} + i\frac{\sqrt{3}}{2}\left(1 - \frac{\delta^2}{9}\right)$$

Group Velocity = v_b + 1/3 slippage rate

Initial Value Problem

SLAC

$$a = \sum_{j=1}^{3} \frac{-i}{\frac{d}{d\lambda} D|_{\lambda=\lambda_j}} \exp(-i\lambda\bar{z})(i\lambda_j^2 a_0 + \lambda_j b_0 + p_0)$$

Initial values of three variables

$$D(\lambda, \delta) = \lambda^3 - \delta\lambda^2 - 1$$

FEL can be triggered by either
- an initial radiation field
- an initial microbunching
- an initial energy modulation

Example:

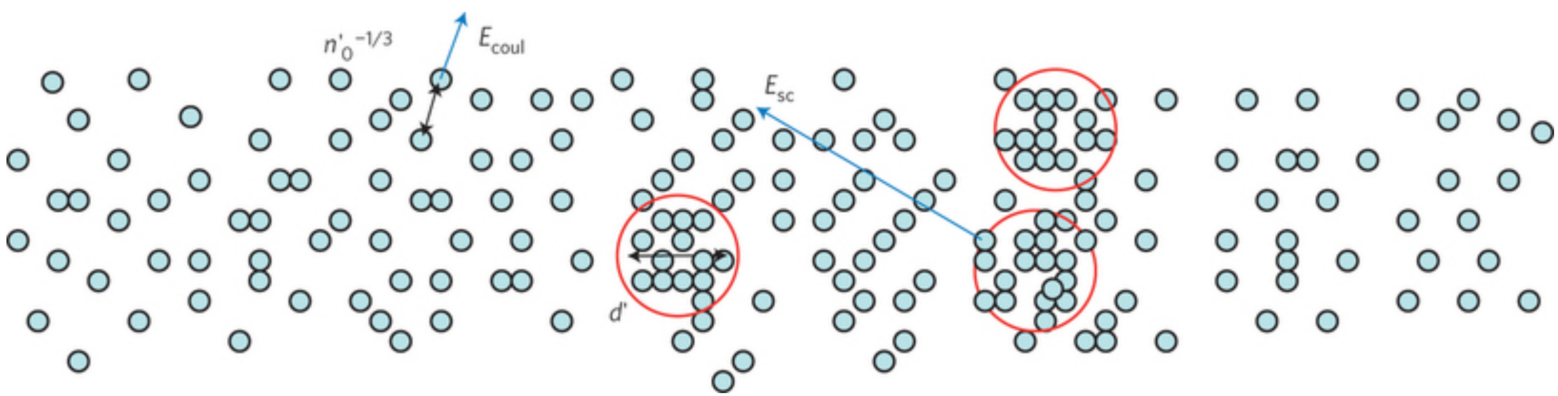
Seeded FEL @ resonance

$$P = \frac{1}{9} P_0 \exp(\sqrt{3}\bar{z})$$

Experimentally, at x-rays it's difficult to generate a starting value for any of these quantities

Shot-Noise

Luckily nature gives us a natural initial value for beam microbunching: NOISE

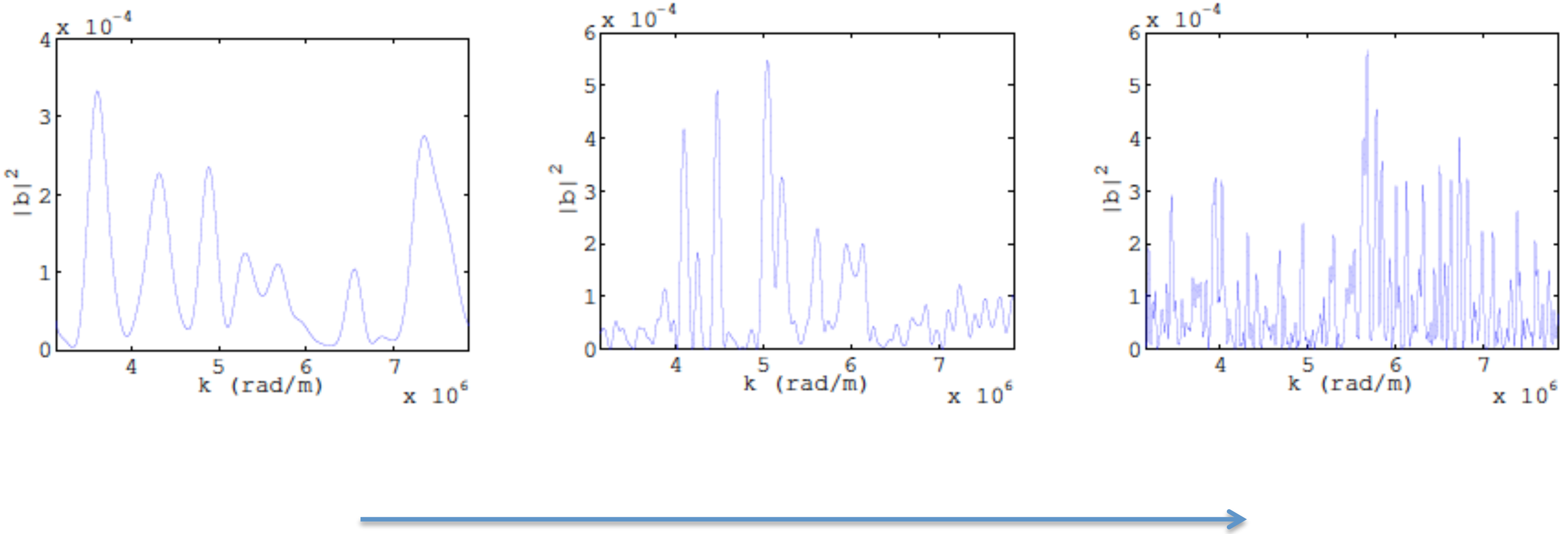


$$< b_{sn} >= 0$$

$$< |b_{sn}|^2 >= \frac{1}{N}$$

Figure from:
Avraham Gover et al.
Nature Physics **8**, 877–880 (2012) doi:10.1038/nphys2443

Shot-Noise Microbunching In Frequency Domain



Increasing bunch length:
Narrower spikes

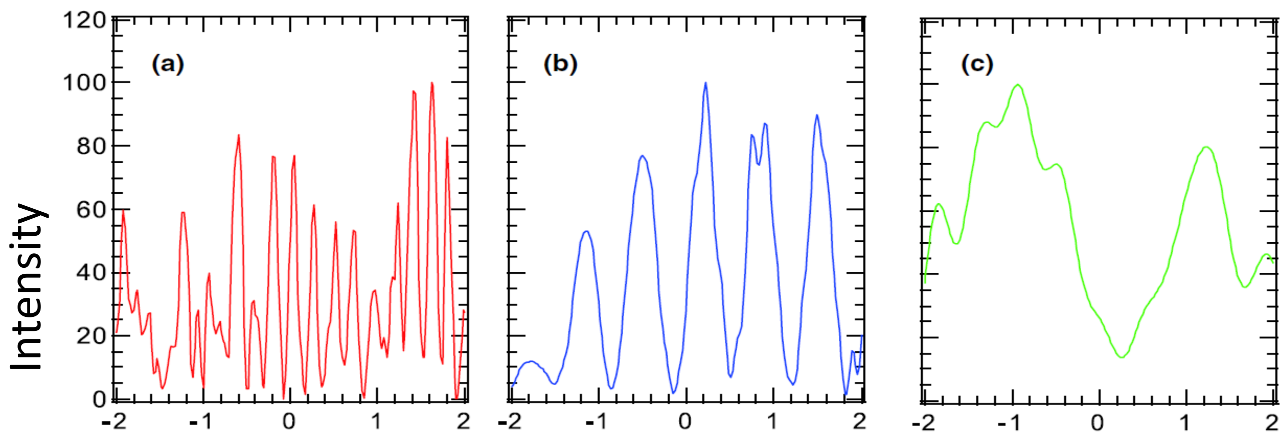
Shot-Noise Microbunching

$$< b(k)b^*(k') >= \frac{1}{N}F(k-k')$$

Spectral autocorrelation ~ Fourier transform of longitudinal distribution at k-k'

(Nice derivation in Saldin's book!)

ΔT = 30 fs   ΔT = 10 fs   ΔT = 4.5 fs



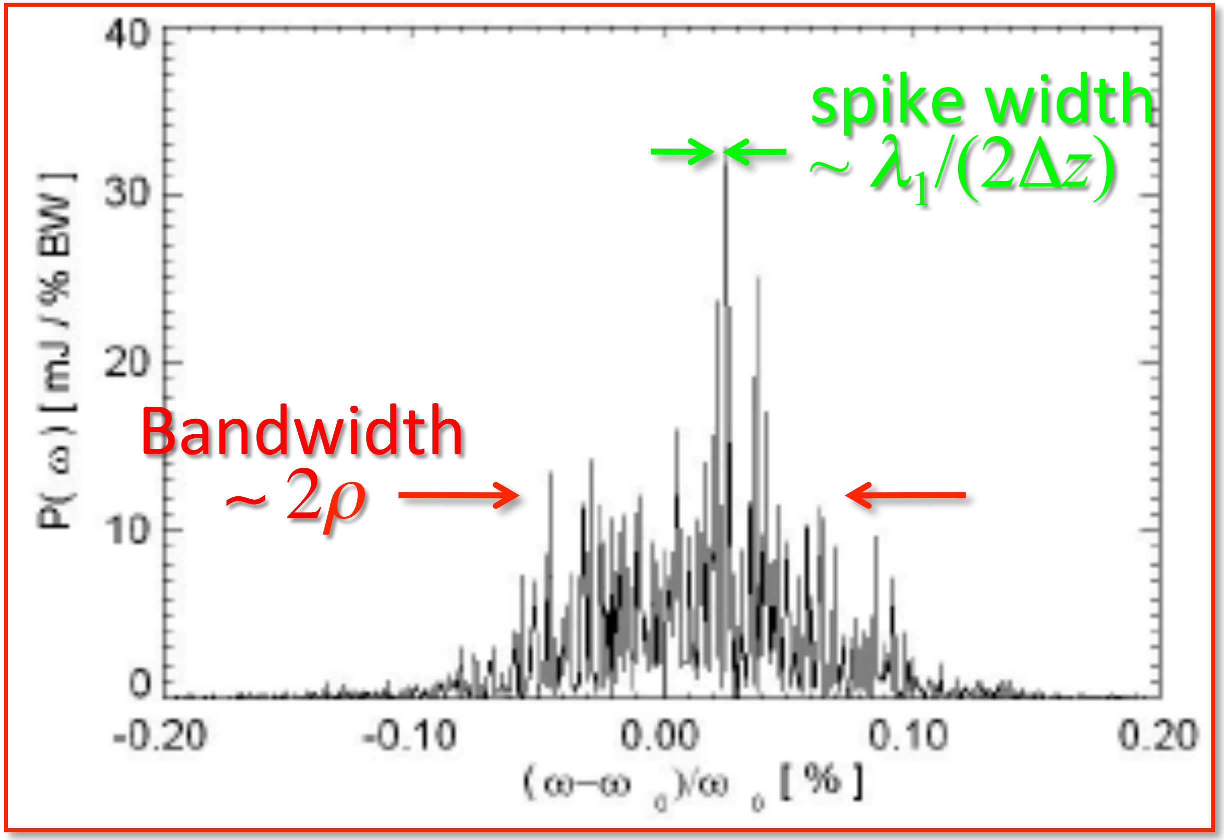
ΔPhoton Energy

PRL 109, 144801 (2012)   PHYSICAL REVIEW LETTERS   week ending 5 OCTOBER 2012

Determination of the Pulse Duration of an X-Ray Free Electron Laser Using Highly Resolved Single-Shot Spectra

Yuichi Inubushi,[1,*] Kensuke Tono,[2] Tadashi Togashi,[2] Takahiro Sato,[1] Takaki Hatsui,[1] Takashi Kameshima,[2] Kazuaki Togawa,[1] Toru Hara,[1] Takashi Tanaka,[1] Hitoshi Tanaka,[1] Tetsuya Ishikawa,[1] and Makina Yabashi[1]

SASE Spectrum



Self Amplified Spontaneous Emission

$$a(\bar{z},\delta) = G(\bar{z},\delta) b_0$$

From initial value problem

$$G(\bar{z},\delta) = \frac{-i}{3\lambda - 2\delta} \exp(-i\lambda\bar{z})$$

In SASE b_0 is shot-noise microbunching

$$< b_{sn} > = 0$$

$$< |b_{sn}|^2 > = \frac{1}{N}$$

What Does SASE Look Like in Time Domain?

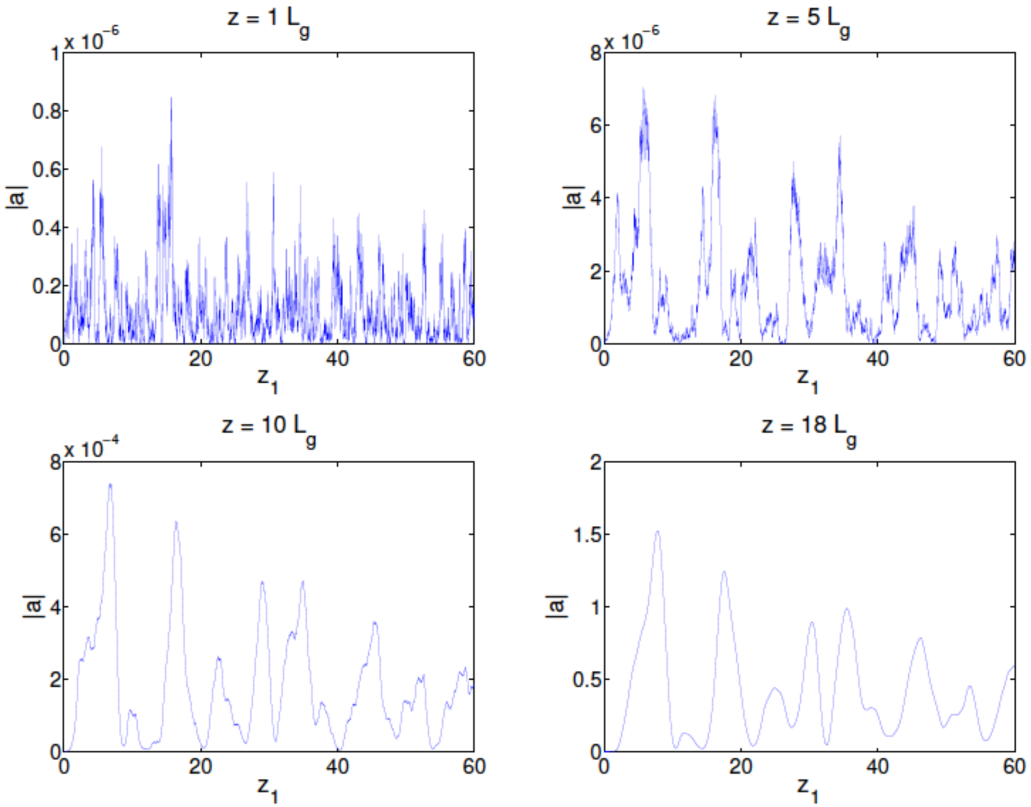
SLAC

$$\bar{a}(\bar{z}, \bar{z}_1) = \frac{k_r L_b \rho}{\pi} \int \exp(i\delta \bar{z}_1) b_0(\delta) G(\bar{z}, \delta) d\delta$$

$$z_1 = (z - v_b t) 2\rho k_r = \zeta / l_c$$

$$l_c = 1/2\rho k_r$$

SLIPPAGE IN 1 GAIN-LENGTH



Spiky temporal structure.

Spikes get broader as radiation slips across the electron bunch!

Using Our 1-D Theory…

SLAC

$$< \bar{a}(\bar{z}_1)\bar{a}^*(\bar{z}_1 + \bar{z}_1') >= \left(\frac{k_r L_b \rho}{N\pi}\right) \int \exp(-i\delta \bar{z}_1') |G(\bar{z}, \delta')|^2 d\delta$$

Wiener's theorem:

Autocorrelation function = Fourier Transform of spectral power density

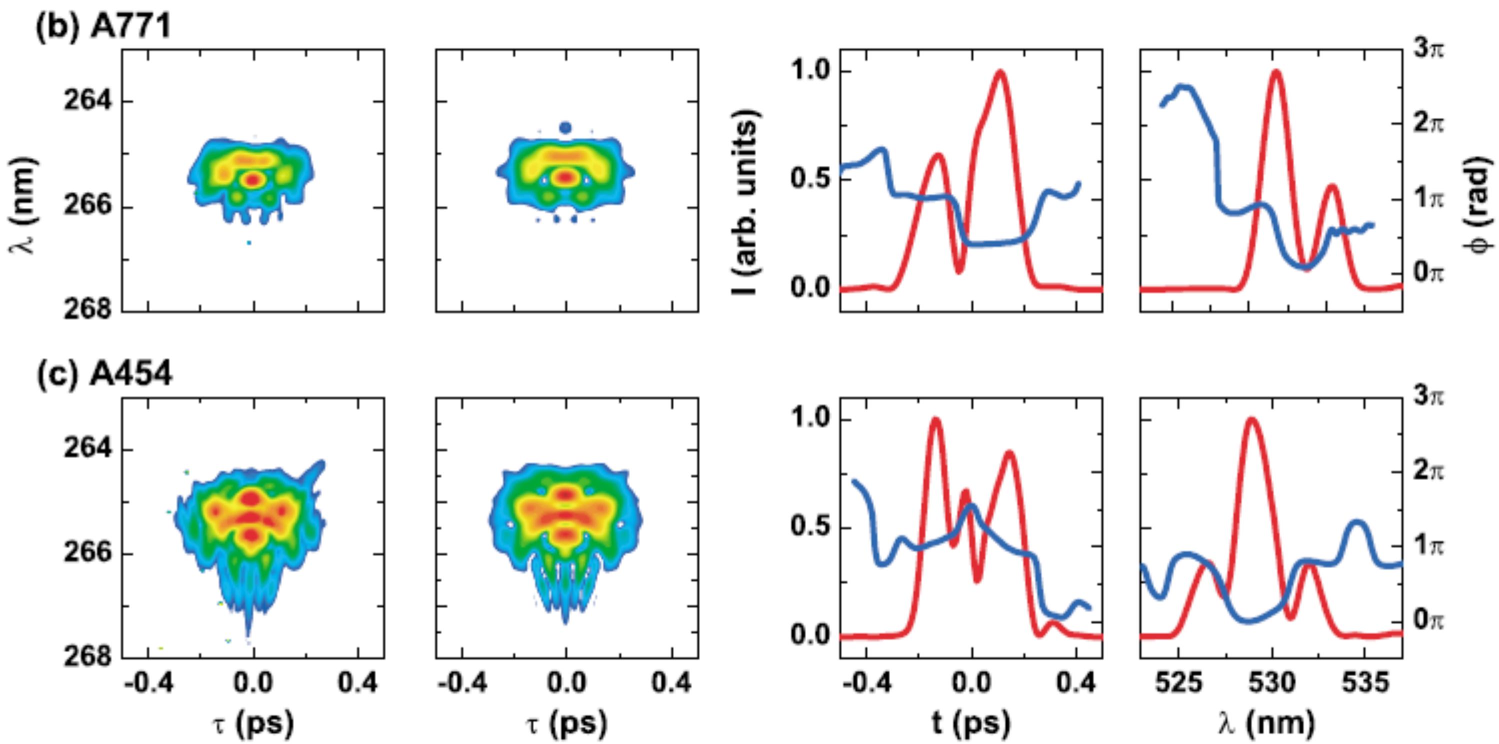
Using the same Gaussian approximation as before:

$$< \bar{a}(\bar{z}, \bar{z}_1)\bar{a}^*(\bar{z}, \bar{z}_1 + \bar{z}_1') >=< |\bar{a}(\bar{z}, \bar{z}_1)|^2 > \exp\left(-\frac{\bar{z}_1'^2}{2\sigma^2_{\bar{z}_1,c}}\right)$$

$$\sigma_{\bar{z}_1,c} = \frac{\sqrt{2\sqrt{3}\bar{z}}}{3}$$

Coherence length grows as a function of time! (Consistently with our intuition from previous slide…)

SASE Spikes: Experimental Observation



VOLUME 91, NUMBER 24 PHYSICAL REVIEW LETTERS week ending 12 DECEMBER 2003

Characterization of a Chaotic Optical Field Using a High-Gain, Self-Amplified Free-Electron Laser

Yuelin Li,[1,*] Samuel Krinsky,[2] John W. Lewellen,[1] Kwang-Je Kim,[1] Vadim Sajaev,[1] and Stephen V. Milton[1]

What is the Average Power?

$$\bar{a}(\bar{z},\bar{z}_1) = \frac{k_r L_b \rho}{\pi} \int \exp(i\delta\bar{z}_1) b_0(\delta) G(\bar{z},\delta) d\delta$$

We can use Parseval's theorem to compute average power

$$< |\bar{a}(\bar{z},\bar{z}_1)|^2 > = \frac{k_r L_b \rho}{N\pi} \int |G(\bar{z},\delta)|^2 d\delta$$

Equivalent Shot-Noise Power

Approximate solution by neglecting d dependence of residue term:
Gain function turns into a Gaussian!

$$P_{SASE} = \frac{1}{9} P_{sn} \exp\left(2\rho k_w \sqrt{3} z\right)$$

$$P_{sn} = P_b \frac{6\rho^2}{N_\lambda} \sqrt{\frac{\pi}{\bar{z}\sqrt{3}}}$$

N_λ number of particles in a wavelength

~few to tens of kW for typical x-ray FELs

Bibliography

COLLECTIVE INSTABILITIES AND HIGH-GAIN REGIME IN A FREE ELECTRON LASER

R. BONIFACIO *, C. PELLEGRINI
National Synchrotron Light Source, Brookhaven National Laboratory, Upton, NY 11973, USA

and

L.M. NARDUCCI
Physics Department, Drexel University, Philadelphia, PA 19104, USA

Received 5 April 1984

VOLUME 73, NUMBER 1 PHYSICAL REVIEW LETTERS 4 JULY 1994

Spectrum, Temporal Structure, and Fluctuations in a High-Gain Free-Electron Laser Starting from Noise

R. Bonifacio,[1,2] L. De Salvo,[1] P. Pierini,[2] N. Piovella,[1] and C. Pellegrini[3]

More Advanced Theories…

So far we have made a number of assumptions

1) No velocity spread:

energy-spread = 0

emittance = 0

2) No diffraction

3) Small signal

4) Slowly varying envelope

More Advanced Theories…

SLAC

So far we have made a number of assumptions

1) No velocity spread:
   energy-spread = 0
   emittance = 0

2) No diffraction

3) Small signal

4) Slowly varying envelope

Increasing levels of complication can address these theoretically

More Advanced Theories…

So far we have made a number of assumptions

1) No velocity spread:
   energy-spread = 0
   emittance = 0

2) No diffraction

3) Small signal

Needed for any reasonble analytical model BUT it obviously fails at saturation…

4) Slowly varying envelope

More Advanced Theories…

SLAC

So far we have made a number of assumptions

1) No velocity spread:
   energy-spread = 0
   emittance = 0

2) No diffraction

3) Small signal

4) Slowly varying envelope

Verified in most cases of interest
(it can be dropped and equations can be solved numerically…)

See Maroli et al.
PHYSICAL REVIEW SPECIAL TOPICS - ACCELERATORS AND BEAMS 14, 070703 (2011)

Fully 3-D Theory

Introduction to the Physics of Free Electron Lasers [1]

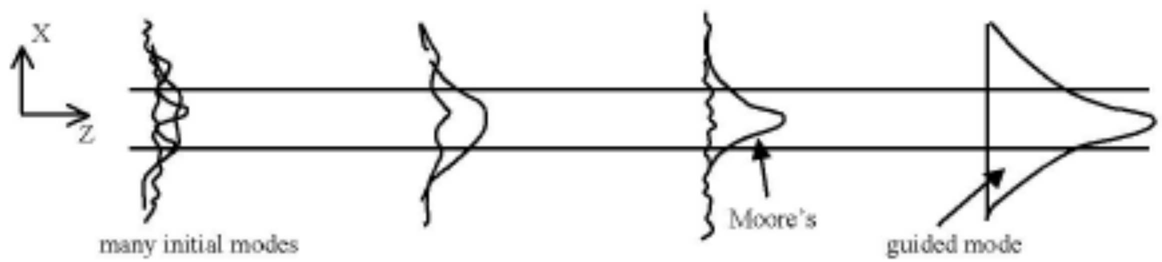
Kwang-Je Kim (ANL), Zhirong Huang (SLAC), Ryan Lindberg (ANL)

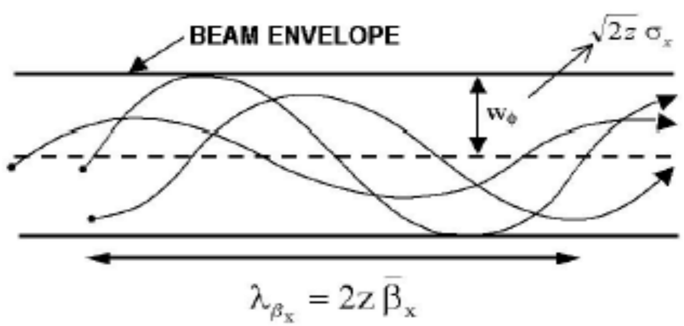
June 10, 2010

Fully 3-D theory that includes:

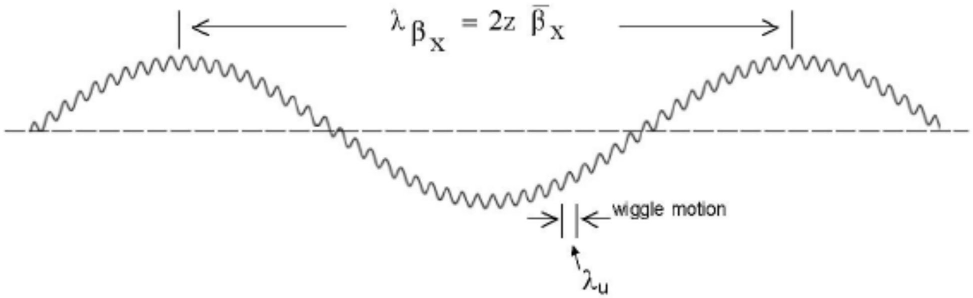
- -e-spread
- -emittance
- -diffraction

Developed by Yu, Kim, Xie and Huang





Fully 3-D Theory



$$x_\beta = x_o \cos(k_{\beta x} z) + \frac{x'_0}{k_{\beta x}} \sin(k_{\beta x} z),$$
$$x'_\beta = x'_o \cos(k_{\beta x} z) - x_0 k_{\beta x} \sin(k_{\beta x} z).$$

Electrons perform transverse betatron oscillations

Introduction to the Physics of Free Electron Lasers [1]

Kwang-Je Kim (ANL), Zhirong Huang (SLAC), Ryan Lindberg (ANL)

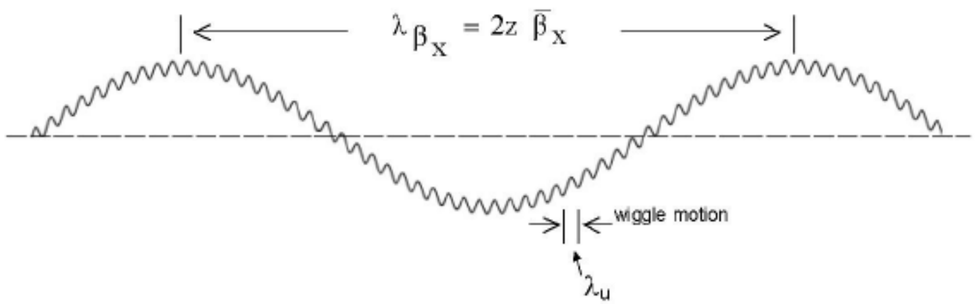
June 10, 2010

Fully 3-D Theory

Fully 3-D theory that includes:

- -e-spread
- -emittance
- -diffraction

Developed by L. H. Yu, Kim, Xie and Hua



$$x_\beta = x_o \cos(k_{\beta x} z) + \frac{x'_0}{k_{\beta x}} \sin(k_{\beta x} z),$$
$$x'_\beta = x'_o \cos(k_{\beta x} z) - x_0 k_{\beta x} \sin(k_{\beta x} z).$$

Electrons perform transverse betatron oscillations

$$\beta_{||} = \sqrt{1 - \frac{1}{\gamma^2} - \beta_\perp^2} \approx 1 - \frac{1}{2\gamma^2} - \frac{\beta_\perp^2}{2}$$

Transverse oscillations introduce an extra term in LONGITUDINAL velocity spread.

$$f = \bar{f}_0 + f_1$$

Density in 6-D phase-space

$$f = \bar{f}_0 + f_1$$

Stationary 0-th order distribution

$$\bar{f}_0(\bar{\boldsymbol{p}}^2 + \bar{k}_\beta^2 \bar{\boldsymbol{x}}^2, \bar{\eta}) = \frac{1}{2\pi \bar{k}_\beta^2 \bar{\sigma}_x^2} \exp\left(-\frac{\bar{\boldsymbol{p}}^2 + \bar{k}_\beta^2 \bar{\boldsymbol{x}}^2}{2\bar{k}_\beta^2 \bar{\sigma}_x^2} \right) \frac{1}{\sqrt{2\pi \bar{\sigma}_\eta}} \exp\left(-\frac{\bar{\eta}^2}{2\bar{\sigma}_\eta^2} \right)$$

$$f = \bar{f}_0 + f_1$$

Perturbation

$$|f_1| << f_0$$

Coupled Linear Equations

$$\begin{pmatrix} \mu_l A_l + \left(-\bar{\nu} + \frac{\bar{\boldsymbol{\nabla}}_\perp^2}{2}\right) A_l + i \int d^2\bar{p} \int d\bar{\eta}\mathcal{F}_l \\ \mu_l \mathcal{F}_l + i A_l \frac{\partial \bar{f}_0}{\partial \bar{\eta}} + \left[-\nu\dot{\theta} + i\left(\bar{\boldsymbol{p}}\frac{\partial}{\partial \bar{\boldsymbol{x}}} - \bar{k}_\beta^2 \bar{\boldsymbol{x}} \frac{\partial}{\partial \bar{\boldsymbol{p}}}\right)\right] \mathcal{F}_l \end{pmatrix} = 0.$$

$$\dot{\theta} = \frac{d\theta}{d\bar{z}} = \bar{\eta} - \frac{\bar{\boldsymbol{p}}^2 + \bar{k}_\beta^2 \bar{\boldsymbol{x}}^2}{2}$$

Coupled Linear Equations

$$\begin{pmatrix} \mu_l A_l + \left(-\bar{\nu} + \frac{\bar{\boldsymbol{\nabla}}_\perp^2}{2}\right) A_l + i \int d^2\bar{p} \int d\bar{\eta} \mathcal{F}_l \\ \mu_l \mathcal{F}_l + i A_l \frac{\partial \bar{f}_0}{\partial \bar{\eta}} + \left[-\nu\dot{\theta} + i \left(\bar{\boldsymbol{p}} \frac{\partial}{\partial \bar{\boldsymbol{x}}} - \bar{k}_\beta^2 \bar{\boldsymbol{x}} \frac{\partial}{\partial \bar{\boldsymbol{p}}}\right)\right] \mathcal{F}_l \end{pmatrix} = 0.$$

Transverse betatron oscillation term

Coupled Linear Equations

$$\begin{pmatrix} \mu_l A_l + \left(-\bar{\nu} + \frac{\bar{\boldsymbol{\nabla}}_\perp^2}{2}\right) A_l + i \int d^2\bar{p} \int d\bar{\eta} \mathcal{F}_l \\ \mu_l \mathcal{F}_l + i A_l \frac{\partial \bar{f}_0}{\partial \bar{\eta}} + \left[-\nu\dot{\theta} + i\left(\bar{\boldsymbol{p}}\frac{\partial}{\partial \bar{\boldsymbol{x}}} - \bar{k}_\beta^2 \bar{\boldsymbol{x}} \frac{\partial}{\partial \bar{\boldsymbol{p}}}\right)\right] \mathcal{F}_l \end{pmatrix} = 0.$$

Coupled Linear Equations

$$\begin{pmatrix} \mu_l A_l + \left(-\bar{\nu} + \frac{\bar{\nabla}_\perp^2}{2}\right) A_l + i \int d^2\bar{p} \int d\bar{\eta} \mathcal{F}_l \\ \mu_l \mathcal{F}_l + i A_l \frac{\partial \bar{f}_0}{\partial \bar{\eta}} + \left[-\nu\dot{\theta} + i\left(\bar{\boldsymbol{p}}\frac{\partial}{\partial \bar{\boldsymbol{x}}} - \bar{k}_\beta^2 \bar{\boldsymbol{x}} \frac{\partial}{\partial \bar{\boldsymbol{p}}}\right)\right] \mathcal{F}_l \end{pmatrix} = 0.$$

Solve distribution function as a function of radiation field

$$\mathcal{F}_l = -\frac{\partial \bar{f}_0}{\partial \bar{\eta}} \int_{-\infty}^{0} d\tau A_l \left(\bar{\boldsymbol{x}}_+\right) e^{i(\nu\dot{\theta} - \mu_n)\tau}$$

$$\bar{\boldsymbol{x}}_+ = \bar{\boldsymbol{x}} \cos \bar{k}_\beta \tau + \frac{\bar{\boldsymbol{p}}}{\bar{k}_\beta} \sin \bar{k}_\beta \tau$$

Coupled Linear Equations

$$\begin{pmatrix} \mu_l A_l + \left(-\bar{\nu} + \frac{\bar{\boldsymbol{\nabla}}_\perp^2}{2}\right) A_l + i \int d^2\bar{p} \int d\bar{\eta} \mathcal{F}_l \\ \mu_l \mathcal{F}_l + i A_l \frac{\partial \bar{f}_0}{\partial \bar{\eta}} + \left[-\nu\dot{\theta} + i\left(\bar{\boldsymbol{p}}\frac{\partial}{\partial \bar{\boldsymbol{x}}} - \bar{k}_\beta^2 \bar{\boldsymbol{x}} \frac{\partial}{\partial \bar{\boldsymbol{p}}}\right)\right] \mathcal{F}_l \end{pmatrix} = 0.$$

Substitute in the 3-D Field Equation

$$\left(\mu - \bar{\nu} + \frac{\bar{\boldsymbol{\nabla}}_\perp^2}{2}\right) A(\bar{\boldsymbol{x}}) - \frac{1}{2\pi\bar{\sigma}_x^2}\int_{-\infty}^{0} \tau d\tau \frac{e^{-\bar{\sigma}_\eta^2\tau^2/2 - i\mu\tau}}{\sin^2(\bar{k}_\beta\tau)} \int d^2\bar{x}' A(\bar{\boldsymbol{x}}') \times \exp\left[-\frac{\bar{\boldsymbol{x}}^2 + (\bar{\boldsymbol{x}}')^2 - 2\bar{\boldsymbol{x}}\cdot\bar{\boldsymbol{x}}'\cos(\bar{k}_\beta\tau)}{2\sin^2(\bar{k}_\beta\tau)}\left(i\bar{k}_\beta^2\tau + \frac{1}{\bar{\sigma}_x^2}\right)\right] = 0.$$

No Panic!

IT LOOKS UGLY BUT…
The dispersion relation can be expressed as a function of 4 dimensionless parameters

$$\left(\mu - \bar{\nu} + \frac{\bar{\nabla}_\perp^2}{2}\right) A(\bar{\boldsymbol{x}}) - \frac{1}{2\pi\bar{\sigma}_x^2}\int_{-\infty}^{0} \tau d\tau \frac{e^{-\bar{\sigma}_\eta^2\tau^2/2 - i\mu\tau}}{\sin^2(\bar{k}_\beta\tau)} \int d^2\bar{x}' A(\bar{\boldsymbol{x}}')$$
$$\times \exp\left[-\frac{\bar{\boldsymbol{x}}^2 + (\bar{\boldsymbol{x}}')^2 - 2\bar{\boldsymbol{x}}\cdot\bar{\boldsymbol{x}}'\cos(\bar{k}_\beta\tau)}{2\sin^2(\bar{k}_\beta\tau)}\left(i\bar{k}_\beta^2\tau + \frac{1}{\bar{\sigma}_x^2}\right)\right] = 0.$$

No Panic!

IT LOOKS UGLY BUT…
The dispersion relation can be expressed as a function of 4 dimensionless parameters

$$\left(\mu - \bar{\nu} + \frac{\bar{\boldsymbol{\nabla}}_\perp^2}{2}\right)A(\bar{\boldsymbol{x}}) - \frac{1}{2\pi\bar{\sigma}_x^2}\int_{-\infty}^{0} \tau d\tau \frac{e^{-\bar{\sigma}_\eta^2\tau^2/2 - i\mu\tau}}{\sin^2(\bar{k}_\beta\tau)}\int d^2\bar{x}' A(\bar{\boldsymbol{x}}')$$

$$\times \exp\left[-\frac{\bar{\boldsymbol{x}}^2 + (\bar{\boldsymbol{x}}')^2 - 2\bar{\boldsymbol{x}}\cdot\bar{\boldsymbol{x}}'\cos(\bar{k}_\beta\tau)}{2\sin^2(\bar{k}_\beta\tau)}\left(i\bar{k}_\beta^2\tau + \frac{1}{\bar{\sigma}_x^2}\right)\right] = 0.$$

$\bar{\nu}$

Detuning / ρ

No Panic!

SLAC

IT LOOKS UGLY BUT…
The dispersion relation can be expressed as a function of 4 dimensionless parameters

$$\left(\mu - \bar{\nu} + \frac{\bar{\boldsymbol{\nabla}}_\perp^2}{2}\right) A(\bar{\boldsymbol{x}}) - \frac{1}{2\pi\bar{\sigma}_x^2}\int_{-\infty}^{0} \tau d\tau \frac{e^{-\bar{\sigma}_\eta^2\tau^2/2 - i\mu\tau}}{\sin^2(\bar{k}_\beta\tau)} \int d^2\bar{x}' A(\bar{\boldsymbol{x}}') \times \exp\left[-\frac{\bar{\boldsymbol{x}}^2 + (\bar{\boldsymbol{x}}')^2 - 2\bar{\boldsymbol{x}}\cdot\bar{\boldsymbol{x}}'\cos(\bar{k}_\beta\tau)}{2\sin^2(\bar{k}_\beta\tau)}\left(i\bar{k}_\beta^2\tau + \frac{1}{\bar{\sigma}_x^2}\right)\right] = 0.$$

$$\bar{\sigma}_x = \sigma_x\sqrt{2k_1k_u\rho} = \frac{2}{\sqrt{3}}\frac{L_R}{L_{G0}}$$

Diffraction negligible if

$$Z_R > L_{G0} \quad \text{or} \quad \bar{\sigma}_x > 1$$

Diffraction parameter

No Panic!

IT LOOKS UGLY BUT…
The dispersion relation can be expressed as a function of 4 dimensionless parameters

$$\left(\mu - \bar{\nu} + \frac{\bar{\boldsymbol{\nabla}}_\perp^2}{2}\right) A(\bar{\boldsymbol{x}}) - \frac{1}{2\pi\bar{\sigma}_x^2} \int_{-\infty}^{0} \tau d\tau \frac{e^{-\bar{\sigma}_\eta^2 \tau^2/2 - i\mu\tau}}{\sin^2(\bar{k}_\beta \tau)} \int d^2\bar{x}' A(\bar{\boldsymbol{x}}') \times \exp\left[-\frac{\bar{\boldsymbol{x}}^2 + (\bar{\boldsymbol{x}}')^2 - 2\bar{\boldsymbol{x}} \cdot \bar{\boldsymbol{x}}' \cos(\bar{k}_\beta \tau)}{2\sin^2(\bar{k}_\beta \tau)} \left(i\bar{k}_\beta^2 \tau + \frac{1}{\bar{\sigma}_x^2}\right)\right] = 0.$$

$$\bar{\sigma}_\eta = \frac{\Delta\gamma}{\gamma\rho}$$

Energy Spread Parameter
(same as 1-D theory!)

No Panic!

IT LOOKS UGLY BUT…

The dispersion relation can be expressed as a function of 4 dimensionless parameters

$$\left(\mu - \bar{\nu} + \frac{\bar{\boldsymbol{\nabla}}_\perp^2}{2}\right) A(\bar{\boldsymbol{x}}) - \frac{1}{2\pi\bar{\sigma}_x^2}\int_{-\infty}^{0} \tau d\tau \frac{e^{-\bar{\sigma}_\eta^2\tau^2/2 - i\mu\tau}}{\sin^2(\bar{k}_\beta\tau)} \int d^2\bar{x}' A(\bar{\boldsymbol{x}}') \times \exp\left[-\frac{\bar{\boldsymbol{x}}^2 + (\bar{\boldsymbol{x}}')^2 - 2\bar{\boldsymbol{x}}\cdot\bar{\boldsymbol{x}}'\cos(\bar{k}_\beta\tau)}{2\sin^2(\bar{k}_\beta\tau)}\left(i\bar{k}_\beta^2\tau + \frac{1}{\bar{\sigma}_x^2}\right)\right] = 0.$$

$$\bar{\sigma}_x^2\bar{k}_\beta = \frac{\varepsilon}{2\varepsilon_r}$$

Emittance parameter

$$\varepsilon_r = \lambda_1/(4\pi)$$

Variational Principle

$$\exp(-wR^2)$$

Project dispersion relation onto a Gaussian mode (~ fundamental mode…).

$$\frac{\mu_0 - \bar{\nu}}{4w} - \frac{1}{4\bar{\sigma}_x^2} = \int_{-\infty}^{0} \frac{\tau d\tau e^{-\bar{\sigma}_\eta^2 \tau^2/2 - i\mu_0 \tau}}{(1 + i\bar{k}_\beta^2 \bar{\sigma}_x^2 \tau)^2 + 4w(1 + i\bar{k}_\beta^2 \bar{\sigma}_x^2 \tau) + 4w^2 \sin^2(\bar{k}_\beta \tau)}.$$

2 unknown quantities
1 equation…
Need another equation!

$$\partial \mu_0 / \partial w = 0$$

Variational principle:
Solution is a stationary point!

Ming Xie Fitting Formula

SLAC

$$L_G = L_{G0}\frac{\sqrt{3}/2}{\mathrm{Im}(\mu_{00})} = L_{G0}(1+\Lambda)$$

Exact and Variational Solutions of
3D Eigenmodes in High Gain FELs

Ming Xie

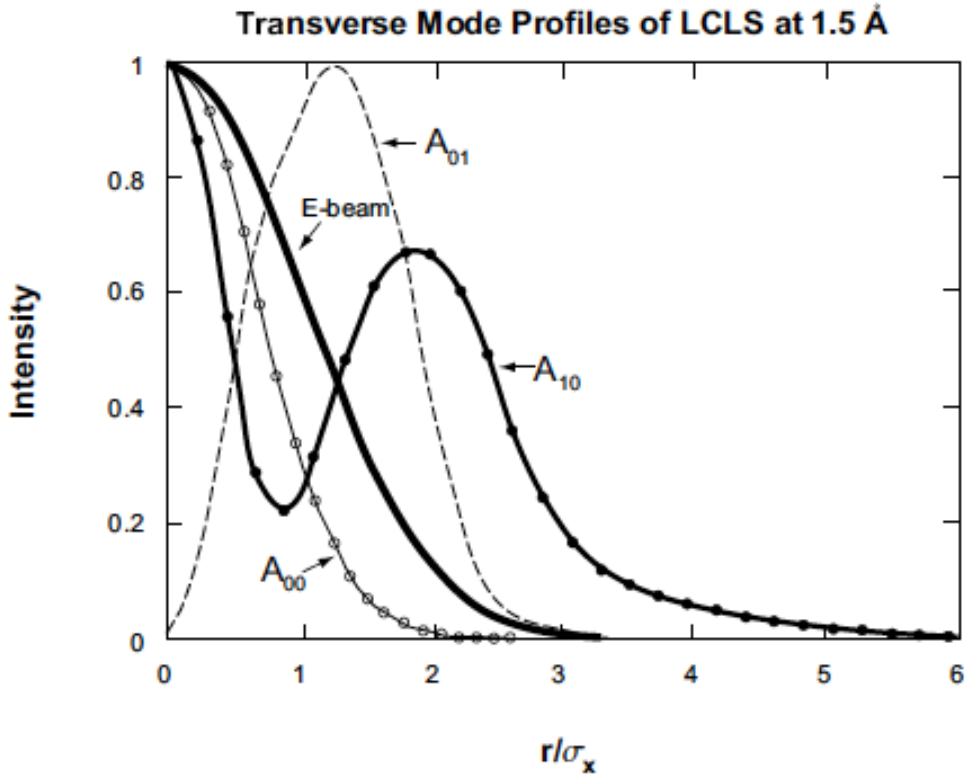
Lawrence Berkeley National Laboratory, Berkeley, California 94720, USA

$$\begin{aligned}\Lambda =& a_1\eta_d^{a_2} + a_3\eta_\varepsilon^{a_4} + a_5\eta_\gamma^{a_6} + a_7\eta_\varepsilon^{a_8}\eta_\gamma^{a_9} \\ &+ a_{10}\eta_d^{a_{11}}\eta_\gamma^{a_{12}} + a_{13}\eta_d^{a_{14}}\eta_\varepsilon^{a_{15}} + a_{16}\eta_d^{a_{17}}\eta_\varepsilon^{a_{18}}\eta_\gamma^{a_{19}}\end{aligned}$$

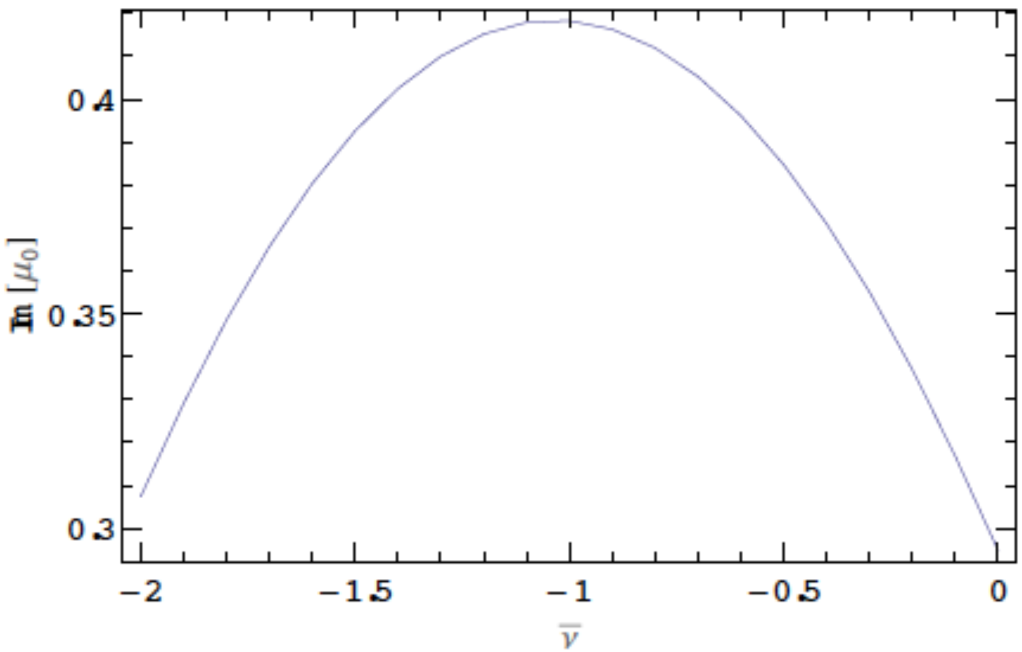
$$\begin{aligned}
a_1 =& 0.45, \quad a_2 = 0.57, \quad a_3 = 0.55, \quad a_4 = 1.6, \quad a_5 = 3, \\
a_6 =& 2, \quad a_7 = 0.35, \quad a_8 = 2.9, \quad a_9 = 2.4, \quad a_{10} = 51, \\
a_{11} =& 0.95, \quad a_{12} = 3, \quad a_{13} = 5.4, \quad a_{14} = 0.7, \quad a_{15} = 1.9, \\
a_{16} =& 1140, \quad a_{17} = 2.2 \quad a_{18} = 2.9, \quad a_{19} = 3.2.
\end{aligned}$$

Examples: LCLS Mode Structure

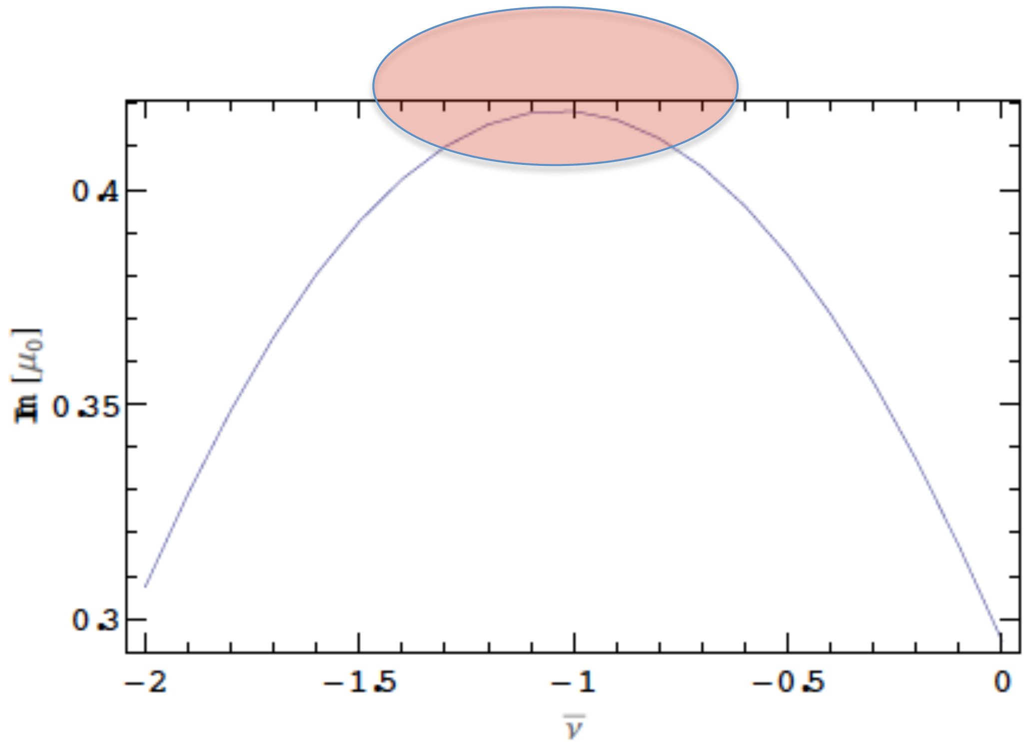
Dispersion relation has to be solved numerically…



Examples: LCLS Detuning Curve



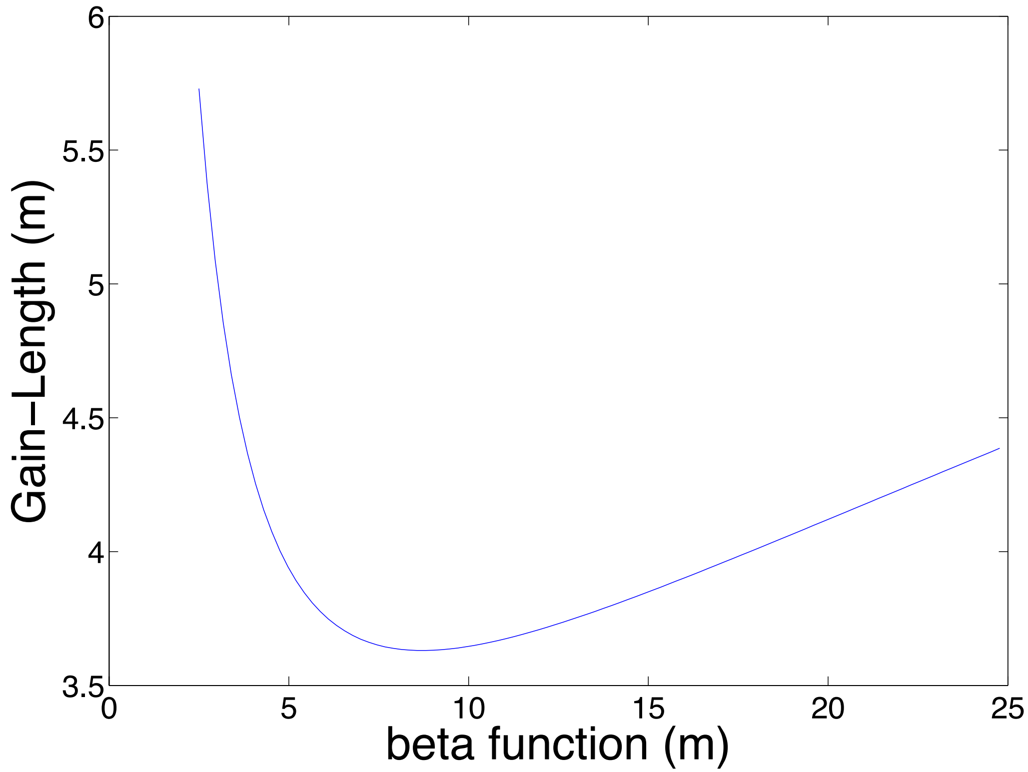
Examples: LCLS Detuning Curve



Note optimum shifts since emittance slows down beam!
(see negative sign!)

$$\dot{\theta} = \frac{d\theta}{d\bar{z}} = \bar{\eta} - \frac{\bar{\boldsymbol{p}}^2 + \bar{k}_\beta^2 \bar{\boldsymbol{x}}^2}{2}$$

Example: Beta Function Optimization



In the 1-D theory the more you focus the shorter the gain-length

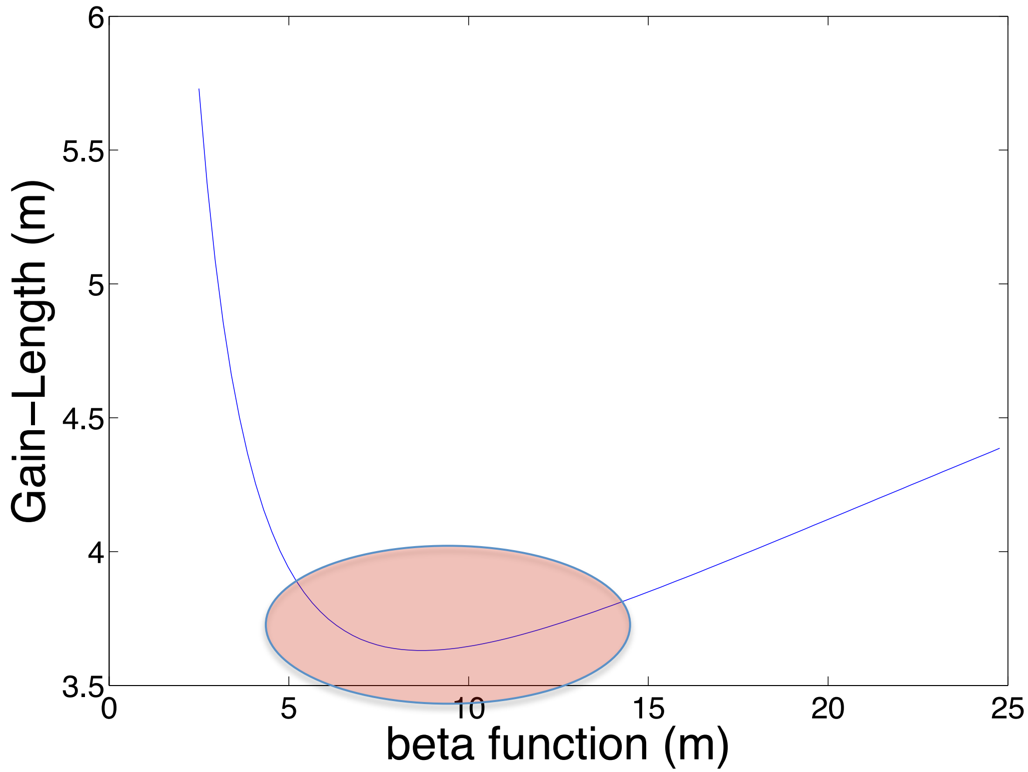
In the 3-D theory you have a competing effect:

$$\sigma_{\beta_\perp}^2 = \varepsilon / \beta\gamma$$

The more you focus the more you increase longitudinal velocity spread!

$$\dot{\theta} = \frac{d\theta}{d\bar{z}} = \bar{\eta} - \frac{\bar{\boldsymbol{p}}^2 + \bar{k}_\beta^2 \bar{\boldsymbol{x}}^2}{2}$$

Example: Beta Function Optimization



In the 1-D theory the more you focus the shorter the gain-length

In the 3-D theory you have a competing effect:

$$\sigma_{\beta_\perp}^2 = \varepsilon / \beta\gamma$$

The more you focus the more you increase longitudinal velocity spread!

$$\dot{\theta} = \frac{d\theta}{d\bar{z}} = \bar{\eta} - \frac{\bar{\boldsymbol{p}}^2 + \bar{k}_\beta^2 \bar{\boldsymbol{x}}^2}{2}$$