

EXTERNAL PARAMETERS CALCULATIONS FOR RF RESONATOR EXCITED OVER AN APERTURE

V.G.Kurakin, Lebedev Physical Institute, Moscow, Russia

Abstract

Bubnov and Galerkin method is used to calculate the coupling strength and cavity detuning for an rf cavity being powered by rf waveguide over a rectangular aperture in cavity surface. Both these so call external parameters are of a value in order to develop properly accelerating system of an accelerator. Electrical fields on coupling aperture surfaces are approximated by finite sums coordinate functions and expansion coefficients are found out by appropriate procedure of solution join of neighboring regions. In Bubnov and Galerkin method, this procedure provides energy flow continuity while crossing coupling surfaces, and this provides fast convergence of appropriate infinite sums. Expressions for aperture conductances are derived followed by formulae for reflection coefficient and cavity detuning.

INTRODUCTION

Usually, partial region technique is used to solve integral or partial differential equation in the region with complicated boundary. For the cavity being excited over an aperture in cavity surface, Maxwell equations can be solved both for waveguide and cavity with the assumption the electric field is known on coupling aperture surface. Integral equation can be written for this field, and different technique may be used to find out approximate solution for the equation. Among various approaches, Bubnov and Galerkin method had been found to be powerful technique in waveguide theory [1,2].

THE GENERAL EXPRESSIONS

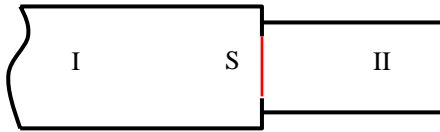


Fig. 1. Cavity excitation

Fig.1 represents schematically the problem under discussion. A cavity (volume II) is powered by a waveguide (volume I) over an aperture s in metallic surface S that separates these volumes. Continuity condition for tangential component of magnetic field

\vec{H} on the aperture may be written as:

$$\left[\vec{H}^I(j, \vec{E}_\tau), \vec{n}_{21} \right] + \left[\vec{H}^{II}(0, \vec{E}_\tau), \vec{n}_{12} \right] = 0, \quad (1)$$

where \vec{E}_τ stands for tangential component of electric field on the aperture and \vec{n} denotes aperture surface normal. According to the technique under discussion one has to search for solution for \vec{E}_τ in the form of coordinate

functions expansion $\vec{E}_\tau = \sum_{l=1}^L e_l \vec{E}_l$, where \vec{E}_l represent aperture coordinate functions. Substituting this sum in (1) one has after multiplying both sides by \vec{E}_m^* and integrating over s :

$$\sum_{l=1}^L e_l Y_{m,l} = h_m, \quad m = 1, 2, \dots, L, \quad (2)$$

where

$$\begin{aligned} Y_{m,l} &= Y_{m,l}^I + Y_{m,l}^{II}, \\ Y_{m,l}^I &= \int_s \vec{E}_m^* [\vec{n}_{21}, \vec{H}^I(0, E_l)] dS, \\ Y_{m,l}^{II} &= \int_s \vec{E}_m^* [\vec{n}_{12}, \vec{H}^{II}(0, E_l)] dS \\ h_m &= \int_s \vec{E}_m^* [\vec{H}^I(j, 0), \vec{n}_{21}] dS \end{aligned} \quad (3)$$

Coordinate function can be chosen as following

$$\vec{E}_e = \nabla_\perp \psi_e; \quad \vec{E}_h = [\nabla_\perp \psi_h, \vec{n}], \quad (4)$$

where functions ψ_e and ψ_h are the solutions of equations:

$$\Delta \psi + \chi^2 \psi = 0 \quad (5)$$

with the boundary conditions

$$\psi_e = 0, \quad \partial \psi_h / \partial n = 0 \quad (6)$$

on on aperture counter.

APERTURE CONDUCTANCES CALCULATION

As it is seen from (3) the magnetic fields distribution induced by electric field on an aperture has to be known in order to calculate partial conductances $Y_{m,l}$. In general, waveguide field is superposition of vector eigenfunctions:

$$\vec{E} = \sum_a U_a \vec{E}_a; \quad \vec{H} = \sum_b I_b \vec{H}_b, \quad (7)$$

where vector eigenfunctions can be found from scalar eigenfunctions ψ satisfying Helmholtz equation (5) with the boundary conditions (6):

$$\vec{E}_{ze} = \psi_e \vec{z}_0; \quad \vec{E}_{qe} = \frac{1}{\chi_e} \nabla_\perp \psi_e; \quad \vec{H}_{qe} = [\vec{z}_0, \vec{E}_{qe}] \quad (8)$$

for TM waves and

$$\vec{H}_{zh} = \psi_h \vec{z}_0; \quad \vec{H}_{qh} = \frac{1}{\chi_h} \nabla_\perp \psi_h; \quad \vec{E}_{qh} = [\vec{H}_{qh}, \vec{z}_0] \quad (9)$$

for TE waves. Supposing vector eigenfunctions to be orthonormal we have the following expressions for amplitude coefficients:

$$U_a = \int_S \vec{E} \vec{E}_a dS; \quad I_a = \int_S \vec{H} \vec{H}_a dS. \quad (10)$$

For a rectangular waveguide with the cross section $a \times b$ scalar eigenfunctions are

$$\begin{aligned} \psi_h &= \sqrt{\frac{(2-\delta_{0m})(2-\delta_{0n})}{ab}} \cos \frac{m\pi}{a} x \cos \frac{n\pi}{b} y \\ \psi_e &= \frac{2}{\sqrt{ab}} \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y \\ \chi_{e,h} &= \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \end{aligned} \quad (11)$$

Suppose that a rectangular cavity is powered by rectangular waveguide over an aperture $l \times d$ in its end. Let us approximate electric field of the aperture with the function

$$\vec{E} = \sqrt{\frac{2}{dl}} \sin \frac{\pi}{l} \left(x - \frac{a-l}{2}\right) \vec{y}_0. \quad (12)$$

In this case one has for waveguide aperture conductance:

$$Y = \sum_a Y_a n_a^2, \quad (13)$$

where

$$n_e = 8 \sqrt{\frac{2}{abdl}} \frac{l}{\pi \chi_{mn}} \frac{\cos \frac{m\pi d}{2a}}{1 - \left(\frac{ml}{a}\right)^2} \sin \frac{m\pi}{2} \sin \frac{n\pi d}{2} \cos \frac{n\pi}{2} \quad (14)$$

$$n_h = 8 \sqrt{\frac{2-\delta_{0n}}{abdl}} \frac{mlb}{n\pi a \chi_{mn}} \frac{\cos \frac{m\pi d}{2a}}{1 - \left(\frac{ml}{a}\right)^2} \sin \frac{m\pi}{2} \sin \frac{n\pi d}{2} \cos \frac{n\pi}{2}$$

It follows from (13) and (14) that

$$Y = Y_{10} n_{10}^2 - \frac{i}{k} \sqrt{\frac{\varepsilon}{\mu}} \frac{128lb}{\pi^4 ad} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{k_{m0}^2}{k_{mn} n^2 \left(1 - \frac{m^2 l^2}{a^2}\right)^2} \times \quad (15)$$

$$\begin{aligned} &\cos^2 \frac{m\pi d}{2a} \sin^2 \frac{n\pi d}{2b} \sin^2 \frac{m\pi}{2} \cos^2 \frac{n\pi}{2} + \\ & - \frac{i}{k} \sqrt{\frac{\varepsilon}{\mu}} \frac{16ld}{\pi^2 ab} \sum_{m=2}^{\infty} \frac{k_{m0}}{\left(1 - \frac{m^2 l^2}{a^2}\right)^2} \cos^2 \frac{m\pi d}{2a} \sin^2 \frac{m\pi}{2} \end{aligned}$$

where

$$\begin{aligned} n_{10}^2 &= \frac{16ld}{\pi^2 ab} \frac{\cos^2 \pi d / 2a}{\left(1 - l^2 / a^2\right)^2}; \\ k_{mn}^2 &= \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2; Y_{10} = \frac{K_{10}}{i\omega\mu} \end{aligned} \quad (16)$$

In the same manner, one arrives at formulae

$$h = 2I_0 (\vec{j}_{ext}) n_{10}, \quad U(\vec{E}_\tau) = en_{10} \quad (17)$$

To calculate partial conductance from resonator volume, one has to use resonator eigenfunctions:

$$\vec{E}_h = \sqrt{\frac{2}{L}} \frac{1}{\chi_h} [\nabla_\perp \psi_h, \vec{z}_0] \sin \frac{l\pi}{L} z \quad (18)$$

$$\vec{H}_h = \sqrt{\frac{2}{L}} \frac{1}{k_h} \left[\frac{1}{\chi_h} \nabla_\perp \psi_h \frac{l\pi}{L} \cos \frac{l\pi}{L} + \chi_h \psi_h \vec{z}_0 \sin \frac{l\pi}{L} z \right] \quad (19)$$

$$l=0,1,2,\dots \quad \vec{E}_e = \sqrt{\frac{2-\delta_{0l}}{L}} \frac{1}{k_e} \left[\frac{1}{\chi_e} \nabla_\perp \psi_e \frac{l\pi}{L} \sin \frac{l\pi}{L} z - \chi_e \psi_e \vec{z}_0 \sin \frac{l\pi}{L} z \right] \quad (20)$$

$$\vec{H}_e = \sqrt{\frac{2-\delta_{0l}}{L}} \frac{1}{\chi_e} [\vec{z}_0, \nabla_\perp \psi_e] \cos \frac{l\pi}{L} \quad (21)$$

$$\vec{E}_g = \sqrt{\frac{2}{L}} \frac{1}{k_g} \left(\nabla_\perp \psi_e \sin \frac{l\pi}{L} z + \psi_e \vec{z}_0 \frac{l\pi}{L} \cos \frac{l\pi}{L} z \right) \quad (22)$$

$$\vec{H}_g = \sqrt{\frac{2-\delta_{0l}}{L}} \frac{1}{k_g} \left(\nabla_\perp \psi_h \cos \frac{l\pi}{L} z - \psi_h \vec{z}_0 \frac{l\pi}{L} \sin \frac{l\pi}{L} z \right)$$

The magnetic field in resonator:

$$\vec{H}(\vec{E}) = \sum_v I_v \vec{H}_v = - \sum_v \frac{i\omega \varepsilon \vec{f}_v}{k_v^2 - k^2} \vec{H}_v \quad (23)$$

and it follows

$$Y = \sum_v \frac{i\omega \varepsilon \vec{f}_v}{k_v^2 - k^2} f_v^2. \quad (24)$$

For our case (rectangular resonator) one can get formula for this conductance immediately keeping in mind the fact that such a resonator is a short cut of a regular waveguide.

$$Y = Y_{10} n_{10}^2 \coth k_{10} B - \frac{i}{k} \sqrt{\frac{\varepsilon}{\mu}} \frac{128lL}{\pi^4 Ad} \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} \frac{k_{m0}^2}{k_{mp} p^2 \left(1 - \frac{m^2 l^2}{A^2}\right)^2} \times \quad (25)$$

$$\begin{aligned} &\cos^2 \frac{m\pi d}{2A} \sin^2 \frac{p\pi d}{2L} \sin^2 \frac{m\pi}{2} \cos^2 \frac{p\pi}{2} \coth k_{mp} B + \\ & - \frac{i}{k} \sqrt{\frac{\varepsilon}{\mu}} \frac{16ld}{\pi^2 AL} \sum_{m=2}^{\infty} \frac{k_{m0}}{\left(1 - \frac{m^2 l^2}{A^2}\right)^2} \cos^2 \frac{m\pi d}{2A} \sin^2 \frac{m\pi}{2} \coth k_{m0} B \end{aligned}$$

All constants have to be taken as for waveguide with the cross section $A \times L$

It is more convenient to represent first term in the last expression in the form as it is appears in formula (24):

$$Y_1 = \frac{i\omega \varepsilon}{k_0^2 - k^2} \frac{2\pi^2 n_{10}^2}{B^3 k_0^2} = Y_{10} n_{10}^2 \coth k_{10} B \quad (26)$$

RESONATOR EXTERNAL PARAMETERS

According (2,17), main wave amplitude in waveguide:

$$U_0^- = \frac{2I_0^+ n_w^2}{Y^w + Y^r} - U_0^+ \quad (27)$$

It follows immediately from this expression, that reflection coefficient at coupling aperture cross section is:

$$\Gamma = \frac{U_0^-}{U_0^+} = \frac{2Y_{10} n_w^2}{Y^w + Y^r} - 1 \quad (28)$$

At resonance the next equivalent conditions take place at aperture cross section:

$$\Gamma_0 = |\Gamma| = \min; \text{Im } \Gamma = 0 \quad (29)$$

For future analysis, we will use the latter. Representing partial conductance's in the form

$$y^w = Y^w / Y_{10} = n_w^2 + ib^w; y^r = Y^r / Y_{10} = y_1 + ib^r \quad (30)$$

where

$$y_1 = \frac{2\pi^2 k^2 n_r^2}{B^3 k_0^3 \hat{K}_{10}} \frac{1}{k_0^2 - k^2}; \hat{K}_{10} = -iK_{10} \quad (31)$$

Taking into account the rf energy being dissipated in the resonator, wave number eigenvalue may be represented in the form:

$$k_0'^2 = k_0^2 + i \frac{k_0^2}{Q}, \quad (32)$$

Q being resonator quality factor. Substituting (32) into (31) one has

$$y_1 = n_r^2 \frac{2\pi^2 k^2}{B^3 \hat{K}_{10} k_0^4} \left\{ \frac{1/Q}{\left(\frac{k_0^2 - k^2}{k_0^2}\right)^2 + \frac{1}{Q^2}} + i \frac{\frac{k_0^2 - k^2}{k_0^2}}{\left(\frac{k_0^2 - k^2}{k_0^2}\right)^2 + \frac{1}{Q^2}} \right\} \quad (33)$$

The last formula can be simplified. Keeping in mind the fact that frequency changing is small and quality factor is much more than unity, one arrives at the expression

$$y_1 = n_r^2 \frac{2\pi^2}{B^3 \hat{K}_{10} k_0^2} \left\{ \frac{1/Q}{(2\Delta k / k_0)^2} \right\} + i \frac{1}{2\Delta k / k_0}, \quad (34)$$

where $\Delta k = k_0 - k$.

Finally, we have the following expressions for the resonator detuning and reflection coefficient as well

$$\frac{\Delta k}{k_0} = - \frac{\pi^2}{B^3 \hat{K}_{10} k_0^2} \frac{n_r^2}{b^w + b^r} \quad (35)$$

$$\Gamma_0 = \frac{1 - \frac{(b^w + b^r)^2}{2\pi^2 Q n_w^2 n_r^2 / B^3 \hat{K}_{10} k_0^2}}{1 + \frac{(b^w + b^r)^2}{2\pi^2 Q n_w^2 n_r^2 / B^3 \hat{K}_{10} k_0^2}} \quad (36)$$

In practice, coupling aperture has the third dimension – thickness, and the whole volume consists of three regions. These are a waveguide (I), a resonator (III) and coupling aperture (II) that in our case can be treated as the waveguide section of the length h with the cross section $l \times d$. Approximate functions

$$\vec{E} = \vec{E}_1 = \vec{E}_2 = \sqrt{\frac{2}{dl}} \sin \frac{\pi}{l} x \vec{y}_0 \quad (37)$$

have to be used for aperture region, an one has the following system

$$\begin{aligned} e_1 Y_{11} + e_2 Y_{12} &= h_1 \\ e_1 Y_{21} + e_2 Y_{22} &= 0 \end{aligned} \quad (38)$$

where

$$Y_{11} = Y^w + Y^a; Y_{22} = Y^{a'} + Y^r \quad (39)$$

$$Y^a = \int_{s1} \vec{E}_1^* [\vec{n}_{21}, \vec{H}^H(0, \vec{E}_1)] dS \quad (40)$$

$$Y^{a'} = \int_{s2} \vec{E}_2 [\vec{n}_{32}, \vec{H}^H(0, \vec{E}_2)] dS$$

$$Y_{12} = \int_{s1} \vec{E}_1^* [\vec{n}_{21}, \vec{H}^H(0, \vec{E}_2)] dS \quad (41)$$

$$Y_{21} = \int_{s2} \vec{E}_2^* [\vec{n}_{23}, \vec{H}^H(0, \vec{E}_1)] dS$$

Since apertures for region II coincide with the full cross section of appropriate waveguide we have (tilde is used for this waveguide parameters)

$$Y^a = Y^{a'} = \tilde{Y}_{10} \coth \tilde{K}_{10} h, \quad (42)$$

and the following expression for reflection coefficient takes place

$$\Gamma = \frac{2Y_{10} n_w^2}{Y_{11} - Y_{12}^2 / Y_{22}} - 1 \quad (43)$$

The same arguments as in the case of "thin" aperture result in the appropriate expressions for the "thick" coupling aperture:

$$\frac{\Delta \omega}{\omega_0} = \frac{\pi^2 n_r^2}{B^3 \hat{K}_{10} k_0^2 b}; b = \frac{b_{12}^2}{b^w + b^r} - (b^r + b^r) \quad (44)$$

$$\Gamma_0 = \frac{1 - g \frac{(b^w + b^r)^2}{b_{12}^2 n_w^2}}{1 + g \frac{(b^w + b^r)^2}{b_{12}^2 n_w^2}}; g = \text{Re } y_1 \quad (45)$$

The dependence of standing wave ratio on aperture dimensions had been measured experimentally. Difference between calculated and measured values didn't exceeded 10%.

CONCLUSION

The method just described proved to be powerful tool to calculate coupling strength. Single function approximation even gives the efficiency quite sufficient for practical applications.

This work had been done at author diploma stage and never before had been published. The author hopes this paper will be useful in practice.

REFERENCES

- [1] S.G.Mikhlin. Variation technique in mathematical physics. Gostechizdat, Moscow, 1957. 476 pp. (In Russian)
- [2] B.M.Mashkovtsev, K.N.Tsybizov, B.F.Emelin. Waveguide theory. Nauka, Moscow, 1966. 352 pp. (In Russian)