ANALYTIC EXPRESSIONS FOR LONGITUDINAL SCHOTTKY SIGNALS FROM BEAMS WITH GAUSSIAN MOMENTUM DISTRIBUTION

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Abstract

We calculate closed analytic expressions for the longitudinal Schottky Signals from beams with gaussian momentum distribution. All dispersion integrals can be evaluated in terms of complex error functions. Using a Padé approximation for the error functions allows very rapid evaluation of the Schottky spectra.

1 INTRODUCTION

In ref. 1 and 2 expression are deduced for the longitudinal Schottky spectrum at the $p-$th harmonic for a beam under the influence of cooling, characterized by the cooling rate ν and a longitudinal impedance $Z_{\|}$

$$P(\Omega) = 2I_0^2 \frac{N}{p} \left| \frac{1 + i\nu I(\Omega)}{\varepsilon(\Omega)} \right|^2 \psi(\Omega/p) \tag{1}$$

where the dielectric function $\varepsilon(\Omega)$ is given by

$$\varepsilon(\Omega) = 1 - iJ(\Omega) \frac{q\eta}{\gamma_0 \beta_0^2} \frac{\omega^2}{2\pi} \frac{I_0 Z_{\|}}{mc^2} + i\nu I(\Omega) \ . \tag{2}$$

The gaussian distribution function $\psi(\omega)$ is defined by

$$\psi(\omega) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\omega-\omega_0)^2}{2\sigma^2}} \ . \tag{3}$$

The dispersion integrals $I(\Omega)$ and $J(\Omega)$ can be evaluated analytically in terms of the complex error function $w(z)$ [3] which is shown in the appendix and are given by

$$\begin{aligned} iI(\Omega) &= i\int_{-\infty}^{+\infty} d\omega \, \frac{\psi(\omega)}{\Omega - p\omega - i\nu} = -\sqrt{\frac{\pi}{2}} \frac{1}{p\sigma} w(z) \quad (4) \\ iJ(\Omega) &= i\int_{-\infty}^{+\infty} d\omega \, \frac{\partial\psi(\omega)/\partial\omega}{\Omega - p\omega - i\nu} = \frac{1}{p\sigma^2}(i - \sqrt{\pi} z w(z)) \end{aligned}$$

with

$$z = -\frac{\Omega - p\omega_0 - i\nu}{\sqrt{2}p\sigma} \tag{5}$$

Having calculated the dispersion integrals in closed form allows us to directly fit eq. 1 to data and extract physical parameters such as frequency spread σ or the damping rate ν from measured schottky spectra. We propose to use the following fitting function

$$\begin{aligned} P(\Omega) &= A + Be^{-\frac{(\Omega-D)^2}{2C^2}} \qquad (6) \\ &\times \left| \frac{1 - \sqrt{\frac{\pi}{2}} E w(z)}{1 - (F + iG)\,(i - \sqrt{\pi} z w(z)) - \sqrt{\frac{\pi}{2}} E w(z)} \right|^2 \end{aligned}$$

with

$$z = -\frac{\Omega - D}{\sqrt{2}C} + \frac{i}{\sqrt{2}} E \ . \tag{7}$$

The fitting parameters $A, B, \ldots, F$ have the following interpretation

A	=	offset or base line
B	=	amplitude of the signal
C	=	frequency spread $p\sigma = p\omega_0\eta\sigma_p$
D	=	center frequency of $p-$th harmonic $p\omega_0$
E	=	scaled damping rate = $\nu/p\sigma$
$F + iG$	=	scaled impedance = $\frac{1}{2\pi}\frac{q\eta}{\gamma_0\beta_0^2}\frac{I_0 Z_{\|}/p}{m_0 c^2}\frac{\omega_0^2}{\sigma^2}$

Note that in this representation the fit-parameter E alone describes the effect of damping and that $F + iG \propto \eta Z_{\|} I_0$ alone describes the effect of the impedance. [1]

2 QUALITATIVE FEATURES

In this section we will exploit eq. 6 and generate longitudinal Schottky spectra that allow us to deduce the beam's properties directly from the shape of spectra similarly to the discussion in ref. 1.

- In the absence of damping ($E = 0$) and collective effects ($F + iG = 0$) the schottky spectrum is given by the momentum distribution $\psi(\Omega/p)$.
- In the absence of impedance ($F + iG = 0$) the cooling rate does not affect the schottky power spectrum as can be seen from eq. 6, where the entire term in absolute values vanishes. Note that the damping rate still enters indirectly, because it affects the equilibrium energy spread and consequently the frequency spread σ as well.
- The effect of cooling is most pronounced at low harmonics as can be seen from the definition of the fit-parameter $E = \nu/p\sigma$.
- A large capacitive impedance ($G > 1$ below transition) causes the Schottky spectrum to exhibt double peaks as can be seen in Fig. 1.
- Adding a resistive impedance ($F < 0$ below transition) causes the low frequency peak to be higher at the expense of the high frequency peak.
- Adding damping, e.g. due to electron cooling, will cause to smooth out the effect of the impedance and

[1] Obviously the same type of parametrisation can also be done for beam transfer functions.

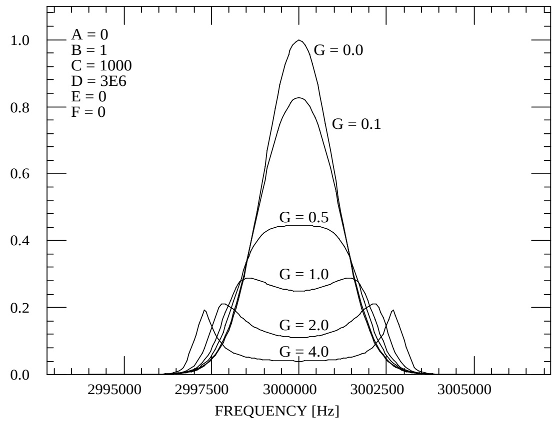

Figure 1: *The effect of a finite capacitive impedance on the longitudinal Schottky spectrum.*

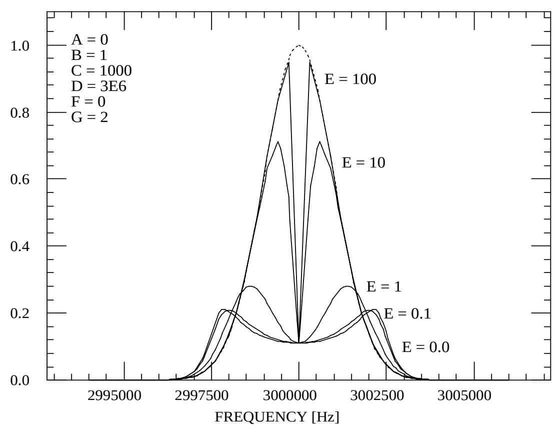

Figure 2: *The effect of damping or cooling on the longitudinal Schottky spectrum in the presence of a finite capacitive impedance (*$G = 2$*). Note that the damping counter-acts the collective effects.*

will make the spectrum more similar to an unperturbed gaussian with a narrow valley in the center. This is illustrated in Fig. 2.

After having discussed the qualitative features of longitudinal Schottky spectra we will discuss the feasability of fitting measured spectra to eq. 6.

3 FITTING SCHOTTKY SPECTRA

In order to test the feasability of determining the seven parameters $A, B, \ldots, G$ from directly fitting eq. 6 to data we generate a longitudinal Schottky spectrum with $A = 0, B = 1, C = 1000, D = 3\,10^6, E = 0, F = -0.1$, and $G = 1$. We then fit eq. 6 to the data by minimizing the cost function

$$\chi^2 = \sum_{\mathrm{data}} (y - P(\Omega; A, B, \ldots, G))^2 \qquad (8)$$

with respect to the parameters $A, B, \ldots, G$. For the minimization a Nelder-Mead Simplex minimizer is used [4].

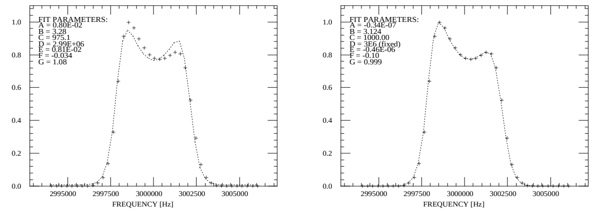

Figure 3: *Fitting a longitudinal Schottky spectrum with* $A = 0, B = 1, C = 1000, D = 3\,10^6, E = 0, F = -0.1, G = 1$. *In the left graph all seven parameters are fitted which results in a rather poor fit and in the right graph the center frequency* D *is fixed at* $3\,10^6$.

In the evaluation of eq. 6 we use a Padé approximation $w(-iz) = P(z)/Q(z)$ for which the coefficients of the polynomial P, Q (of order 10 and 11) are all real [5].

The left graph in Fig. 3 shows the result from fitting all seven parameters simultaneously. Clearly the result is rather poor. In the right graph we fit the same data but fixing the central frequency to its correct value $D = 3\,10^6$ and then fit the remaining six parameters which results in much more accurate results. In practice the central frequency is always known before-hand, because it is a harmonic of the revolution frequency. We presume that the inaccuracy shown in fitting all parameters simultaneously is caused by a correlation between finding the resistive part of the impedance F which causes the asymmetry and the central frequency D.

Finally we analyze Schottky spectra from 2 mA cooled 436 MeV deuterons stored in CELSIUS [6], observed at the 32nd revolution harmonic. From the fit shown in Fig. 4 we deduce a frequency spread of about 1.2 kHz which translates to a momentum spread of $3.5\,10^{-5}$. From the fit parameters F and G we deduce an impedance of $Z_{\|}/p = (0.2 - 2.3i)\,\mathrm{k}\Omega$ which is consistent with earlier measurements. The fitted damping coefficient E, however is much too large and would imply a damping time on the order of ms which is orders of magnitude smaller than expected.

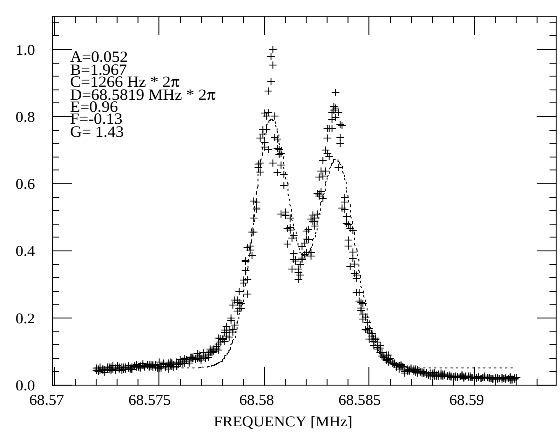

Figure 4: *Schottky spectrum from 436 MeV deuterons.*

Qualitatively, however, the deep narrow valley in spectrum indicates strong damping.This is currently an unresolved problem, which may e.g. be due to a non-gaussian beam or finite reolution bandwidth of the spectrum analyzer.

4 ACKNOWLEDGMENTS

The author wishes to thank D. Reistad, TSL for fruitful discussions.

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A EVALUATION OF $I(\Omega)$

In this appendix we will evaluate the first the dispersion integrals given by eq. 4. If the damping rate ν is positive, we can represent the denominator by

$$\frac{1}{x}=i\int_0^{+\infty}d\alpha\, e^{-i\alpha x} \tag{9}$$

where $\nu>0$ implies $\operatorname{Im}x<0$ which makes the integral convergent. In the limit $\operatorname{Im}x\to 0$ we revover the well known principal value relation

$$\lim\frac{1}{y+i\varepsilon}=\mathrm{PP}\frac{1}{y}-i\pi\delta(y)\;. \tag{10}$$

This trick we use to turn the denominator of eq. 4 into an exponential. Exchanging the order if integration, performing the gaussian integral over ω, and introducing auxiliary variables $x=\omega/\sigma$ and $x_0=\omega_0/\sigma$ we arrive at

$$iI(\Omega)=-\int_0^{+\infty}d\alpha\, e^{-\alpha^2p^2\sigma^2/2-i\alpha(\Omega-p\omega_0-i\nu)}\;. \tag{11}$$

Substituting $\beta=\alpha p\sigma/\sqrt{2}$ we obtain

$$iI(\Omega)=-\frac{\sqrt{2}}{p\sigma}\int_0^{+\infty}d\beta\, e^{-\beta^2+2i\beta z} \tag{12}$$

with z given by eq. 5 in the main text. The integral appearing in eq. ? is a representation of the complex error function $w(z)$ [3]

$$w(z)=\frac{2}{\sqrt{\pi}}\int_0^{+\infty}d\beta\, e^{-\beta^2+2i\beta z}\;. \tag{13}$$

Inserting in eq. ? we finally get the second half the first of eq. 4.

B EVALUATION OF $J(\Omega)$

In this appendix we will evaluate the second dispersion integral in eq. 4 for which we need the derivative of the gaussian distribution function

$$\frac{\partial\psi(\omega)}{\partial\omega}=-\frac{1}{\sqrt{2\pi}\sigma}\frac{\omega-\omega_0}{\sigma^2}e^{-(\omega-\omega_0)^2/2\sigma^2} \tag{14}$$

Inserting in eq. 4 and again turning the denominator into an exponential with the aid of eq. ? we can exchange the order of integration and get

$$\begin{aligned} iJ(\Omega) &= \frac{1}{\sqrt{2\pi}\sigma^3}\int_0^{+\infty}d\alpha\, e^{-i\alpha(\Omega-i\nu)} \\ &\times \int_{-\infty}^{+\infty}d\omega\,(\omega-\omega_0)e^{-(\omega-\omega_0)^2/2\sigma^2+i\alpha p\omega}\;. \end{aligned} \tag{15}$$

The ω appears linearly in the integral and is taken care of by parametric differentiation which can be pulled out of the $d\omega$ integral. Remembering that the $\partial/\partial\alpha$ operator acts to the right hand side only we rewrite the previous equation

$$\begin{aligned} iJ(\Omega) &= \frac{1}{\sqrt{2\pi}\sigma^3}\int_0^{+\infty}d\alpha\, e^{-i\alpha(\Omega-i\nu)} \\ &\times \left(\frac{1}{ip}\frac{\partial}{\partial\alpha}-\omega_0\right)\int_{-\infty}^{+\infty}d\omega\, e^{-(\omega-\omega_0)^2/2\sigma^2+i\alpha p\omega} \end{aligned} \tag{16}$$

which leaves a gaussian integral over $d\omega$ that can be easily evaluated. After performing the differentiation with respect to α we arrive at

$$iJ(\Omega)=ip\int_0^{+\infty}d\alpha\,\alpha e^{-\alpha^2p^2\sigma^2/2-i\alpha(\Omega-p\omega_0-i\nu)} \tag{17}$$

which upon substituting $\beta=\alpha p\sigma/\sqrt{2}$ transforms to

$$iJ(\Omega)=\frac{2i}{p\sigma^2}\int_0^{+\infty}d\beta\,\beta e^{-\beta^2+2iz\beta} \tag{18}$$

with z defined in eq. ?. The β linear in the integral can be treated again by parametric differentiation with the result

$$iJ(\Omega)=\frac{1}{p\sigma^2}\frac{\partial}{\partial z}\int_0^{+\infty}d\beta\, e^{-\beta^2+2iz\beta}=\frac{\sqrt{\pi}}{2p\sigma^2}w'(z) \tag{19}$$

where we use the integral representation of the error function given in eq. ?. Note that $w'(z)$ is the derivative of the complex error function which can be evaluated [3] as

$$w'(z)=\frac{2i}{\sqrt{\pi}}-2zw(z)\;. \tag{20}$$

Thus it suffices to know the complex error function at z in order to calculate its derivative. Utilizing this we arrive at the expression stated in the main part of the text.