EMITTANCE GROWTH AND PARTICLE DIFFUSION INDUCED BY DISCRETE-PARTICLE EFFECTS IN INTENSE BEAMS

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Abstract

We analyze particle diffusion and emittance growth induced by discrete-particle effects in two-dimensional self-consistent numerical simulation studies of beam dynamics. In particular, an analytical model is presented which describes the slow time-scale variation of edge emittance for a perfectly matched beam in a periodic solenoidal magnetic focusing field. A scaling law for edge emittance growth is obtained.

1 DISCUSSION

There has been a growing interest in the study of high-current electron and ion accelerators for a variety of applications. An important issue in the development of such advanced accelerators is to avoid beam halos and associated beam losses [1]. While modern accelerator design relies heavily on self-consistent computer simulations, accurate predictions of the processes of beam halo formation and beam losses have not been accessible in the simulations because of discrete-particle effects [2]. In this paper, we derive a scaling law which governs the processes of edge emittance growth and particle diffusion induced by discrete-particle effects in self-consistent simulations of periodically focused intense charged-particle beams.

Let us consider a thin, continuous charged-particle beam which propagates with average axial velocity $\beta_b c\vec{e}_z$ through an axisymmetric linear focusing channel provided by a periodic solenoidal magnetic field

$$\vec{B}_0(x,y,s) = B_z(s)\,\vec{e}_z - [B_z'(s)/2]\,(x\vec{e}_x + y\vec{e}_y). \quad (1)$$

In Eq. (1) $s = z = \beta_b ct$ is the axial coordinate, $B_z(s+S) = B_z(s)$ is the axial component of the applied magnetic field, S is the fundamental periodicity length of the focusing field, c is the speed of light in $vacuo$, and the "prime" denotes derivative with respect to s.

In the present two-dimensional macroparticle model, the beam density is given by

$$n(x,y,s) = \frac{N}{N_p}\sum_{i=1}^{N_p}\delta[x - x_i(s)]\,\delta[y - y_i(s)], \quad (2)$$

where N and N_p are the number of microparticles and macroparticles per unit axial length of the beam, respectively, and (x_i, y_i) is the transverse displacement of the ith macroparticle from the beam axis at $(x,y) = (0,0)$. Under the paraxial approximation, we can express the transverse equations of motion for the ith macroparticle of the beam in the Larmor frame as [1]

$$\frac{d^2\tilde{x}_i}{ds^2} + \kappa_z(s)\,\tilde{x}_i = -\frac{q}{\gamma_b^3\beta_b^2 mc^2}\frac{\partial}{\partial\tilde{x}_i}\Phi^{(s)}(\tilde{x}_i,\tilde{y}_i,s), \quad (3)$$

$$\frac{d^2\tilde{y}_i}{ds^2} + \kappa_z(s)\,\tilde{y}_i = -\frac{q}{\gamma_b^3\beta_b^2 mc^2}\frac{\partial}{\partial\tilde{y}_i}\Phi^{(s)}(\tilde{x}_i,\tilde{y}_i,s). \quad (4)$$

In Eqs. (3) and (4), $i = 1, 2, ..., N_p$, $\gamma_b = (1 - \beta_b^2)^{-1/2}$ is the relativistic mass factor, m and q are the particle rest mass and charge, respectively, $\kappa_z(s) = [qB_z(s)/2\gamma_b\beta_b mc^2]^2$ is a measure of the strength of the focusing field, and

$$\Phi^{(s)}(\tilde{x}_i,\tilde{y}_i,s) = -\frac{qN}{N_p}\sum_{j=1(j\neq i)}^{N_p}\ln[(\tilde{x}_i-\tilde{x}_j)^2+(\tilde{y}_i-\tilde{y}_j)^2] \quad (5)$$

is the self-field scalar potential associated with the beam space-charge.

In order to develop an analytical model to describe diffusive behavior induced by discrete-particle effects in beam dynamics, we first consider the limit of a smooth equilibrium distribution of particles corresponding to the Kapchinskij-Vladimirskij (KV) equilibrium function [1]. In the KV equilibrium, the beam density is given by

$$n_{KV}(\tilde{x},\tilde{y},s) = \begin{cases} N/\pi r_b^2(s), & 0 \le r \le r_b(s), \\ 0, & r > r_b(s), \end{cases} \quad (6)$$

where $r \equiv (x^2+y^2)^{1/2} = (\tilde{x}^2+\tilde{y}^2)^{1/2}$ is the radial coordinate, and $r_b = r_b(s)$ is the beam radius. The scalar potential for the self-electric field is given by

$$\Phi_{KV}^{(s)}(\tilde{x},\tilde{y},s) = -qNr^2/r_b^2(s) \quad (7)$$

in the beam interior $(r < r_b)$. Substituting $\Phi^{(s)}(\tilde{x}_i,\tilde{y}_i,s) = \Phi_{KV}^{(s)}(\tilde{x}_i,\tilde{y}_i,s)$ into Eqs. (3) and (4), the equilibrium particle orbits $\tilde{x}_i(s)$ and $\tilde{y}_i(s)$ can be expressed as

$$\tilde{x}_i(s) = A_{xi}\,r_b(s)\cos[\psi(s)+\phi_{xi}], \quad (8)$$

$$\tilde{y}_i(s) = A_{yi}\,r_b(s)\sin[\psi(s)+\phi_{yi}], \quad (9)$$

where A_{xi}, $A_{yi} = (1 - A_{xi}^2)^{1/2}$, ϕ_{xi} and ϕ_{yi} are constants determined by the initial conditions, $\psi(s) = 4\epsilon\int_0^s ds/r_b^2(s)$ is the accumulated phase of the betatron oscillations, and $r_b(s) = r_b(s+S)$ solves the beam envelope equation

$$r_b'' + \kappa_z(s)\,r_b - K/r_b - (4\epsilon)^2/r_b^3 = 0, \quad (10)$$

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with ϵ being the unnormalized rms emittance of the beam, and $K \equiv 2q^2N/\gamma_b^3\beta_b^2mc^2$, the normalized perveance of the beam. The particle distribution function for the KV equilibrium can be expressed as $f_{KV}(\tilde{x},\tilde{y},\tilde{x}',\tilde{y}',s) = (N/16\epsilon^2\pi^2)\,\delta(A_x^2+A_y^2-1)$, where $\delta(x)$ is the Dirac δ-function. Because the four-dimensional phase-space volume element is given by $d\tilde{x}d\tilde{y}d\tilde{x}'d\tilde{y}' = 16\epsilon^2 A_x\,A_y\,dA_x dA_y d\phi_x d\phi_y$, integrating f_{KV} over A_y, ϕ_x and ϕ_y yields the distribution function for A_x over a KV beam

$$F_{KV}(A_x) = \begin{cases} 2NA_x, & 0 \le A_x \le 1, \\ 0, & A_x > 1, \end{cases} \qquad (11)$$

where $\int_0^\infty F_{KV}(A_x)dA_x = N$. Note from Eq. (11) that the largest concentration of particles occurs at $A_x = 1$. Note also from Eq. (8) that particles with $A_{xi} = 1$ reach the edge of the beam with $x_i = r_b$, as they execute betatron oscillations. Therefore, they are most likely to leave the beam core under the perturbations induced by discrete-particle effects.

In numerical simulations as well as in experiments, the beam density deviates from the smooth beam density $n_{KV}(\tilde{x},\tilde{y},s)$ of the KV equilibrium. For a coarse-grained uniform density distribution, the deviation is small when there is a large number of particles. Such small deviation will induce slow-time-scale evolution of $A_{xi}(s)$, $A_{yi}(s)$, $\phi_{xi}(s)$ and $\phi_{yi}(s)$ in the particle orbit given in Eqs. (8) and (9). In the remainder of this paper, we analyze the dynamics of edge particles initially with $A_{xi}(s{=}0) = 1$ and $A_{yi}(s{=}0) = [1 - A_{xi}^2(s{=}0)]^{1/2} = 0$, because they are most likely to diffuse away from the beam core as discussed previously. We disregard dynamical couplings between $(A_{xi};\phi_{xi})$ and $(A_{yi};\phi_{yi})$ because $A_{yi}(s) \approx 0$, and introduce the dimensionless variables and parameters defined by $s/S \to s$, $\tilde{x}/(4\epsilon S)^{1/2} \to \tilde{x}$, $\tilde{y}/(4\epsilon S)^{1/2} \to \tilde{y}$, $r_b/(4\epsilon S)^{1/2} \to r_b$, $S^2\kappa_z \to \kappa_z$ and $SK/4\epsilon \to K$. Substituting Eq. (8) into Eq. (3), and taking into account the *slow* dependence of A_{xi} and ϕ_{xi}, we find that

$$\begin{aligned} &\left[A_{xi}'\,r_b' - \frac{A_{xi}\,\phi_{xi}'}{r_b}\right]\cos(\psi+\phi_{xi}) \\ -&\left[\frac{A_{xi}'}{r_b} + A_{xi}\,\phi_{xi}'\,r_b'\right]\sin(\psi+\phi_{xi}) = \\ &-\frac{K}{4qN}\frac{\partial}{\partial x_i}[\phi^{(s)} - \phi_{KV}^{(s)}], \end{aligned} \qquad (12)$$

where use has been made of Eq. (10). It is evident in Eq. (12) that $A_{xi}' = 0 = \phi_{xi}'$ for $\phi^{(s)} = \phi_{KV}^{(s)}$.

To derive a closed set of equations for the slowly varying variables A_{xi} and ϕ_{xi}, we average Eq. (12) over fast oscillations pertaining to the focusing field and the betatron oscillations. Making use of Eqs. (5), and (7)-(9), we can express Eq. (12) as

$$\frac{dA_{xi}}{ds} = -\frac{K}{N_p}\sum_{j=1(j\neq i)}^{N_p}\frac{B_jb_j + C_jc_j}{b_j^2+c_j^2}, \qquad (13)$$

$$\frac{d\phi_{xi}}{ds} = \frac{K}{2} - \frac{K}{A_{xi}N_p}\sum_{j=1(j\neq i)}^{N_p}\frac{C_jb_j - B_jc_j}{b_j^2+c_j^2}, \qquad (14)$$

where

$$B_j = -(A_{xj}/2)\sin\Delta_{xj},\; C_j = (A_{xi} - A_{xj}\cos\Delta_{xj})/2,$$

$$\begin{aligned} b_j = [&(A_{xi} - A_{xj}\cos\Delta_{xj})^2 - A_{xj}^2\sin^2\Delta_{xj} \\ &-A_{yj}^2\cos(2\Delta_{yj})]/2, \end{aligned} \qquad (15)$$

$$c_j = (A_{xi} - A_{xj}\cos\Delta_{xj})A_{xj}\sin\Delta_{xj} + \frac{1}{2}A_{yj}^2\sin(2\Delta_{yj}),$$

with $\Delta_{xj} \equiv \phi_{xj} - \phi_{xi}$ and $\Delta_{yj} \equiv \phi_{yj} - \phi_{xi}$. Since the derivation of Eqs. (13) and (14) does not require the explicit form of the focusing magnetic field $B_z(s)$, Eqs. (13) and (14) are valid for an arbitrary periodic focusing channel.

In principle, detailed dynamics of edge particles initially with $A_{xi} = 1$ and $A_{yi} = 0$ can be analyzed using Eqs. (13) and (14). In this paper, however, we examine particle diffusion induced by discrete-particle effects. To describe the diffusion process quantitatively, we introduce the quantities $\mu(s) = \langle A_{xi}\rangle$ and $\sigma^2(s) = \langle (A_{xi}-\mu)^2\rangle$, where $\langle\;\rangle$ stands for the average over particles that are initially located at $A_{xi} = 1$. We compute the expectation values of $d\mu/ds = \langle A_{xi}'\rangle$ and $d^2\sigma^2/ds^2 = \langle (A_{xi}' - \mu')^2\rangle$ by ensemble averaging over all possible beam distributions which approach the KV distribution when $N_p \to \infty$. The results are $\mu(s) = \mu(0) = 1$, and

$$\sigma^2(s) = D\,s^2, \qquad (16)$$

where the ‘diffusion’ coefficient is given by

$$D(K,N_p) = \overline{\xi}\,K^2/N_p, \qquad (17)$$

$\overline{\xi} = (1/N)\int \xi_j f_{KV}(\tilde{x}_j,\tilde{y}_j,\tilde{x}_j',\tilde{y}_j',s)d\tilde{x}_jd\tilde{y}_jd\tilde{x}_j'd\tilde{y}_j'$, $\xi_j = [(B_jb_j + C_jc_j)/(b_j^2+c_j^2)]^2$. It should be stressed that unlike usual diffusive processes, the variance σ^2 here is proportional to s^2. Due to the highly oscillatory nature of ξ_j, our best estimate of the value of $\overline{\xi}$ is $\overline{\xi} = 0.7 \pm 0.3$. In dimensional units, it follows from Eq. (16) that the edge emittance 4ϵ evolves according to

$$\langle 4\epsilon(s)\rangle = 4\epsilon(0)[1 + \overline{\xi}K^2s^2/16\epsilon^2(0)N_p]. \qquad (18)$$

To verify the scaling law in Eqs. (16) and (17), we carry out self-consistent simulations by integrating Eqs. (3) and (4) numerically for various particle distributions. We adopt the following procedure to calculate the diffusion about $A_{xi} = 1$. In such a self-consistent, a first set of N_p macroparticles is loaded corresponding to a KV distribution, a second set of N_t test particles is loaded at $A_{xi} = 1$ with a uniform distribution of ϕ_{xi} in the range from 0 to 2π. As the beam propagates through the focusing channel, the particles in the first set interact with each other self-consistently, whereas the test particles experience the electric and magnetic forces imposed by the particles in the first set. Integrating Eq. (10) concurrently in the simulation and using the relation $A_{xi} = [(\tilde{x}_i/r_b)^2 + (\tilde{x}_i\, r_b' - \tilde{x}_i'\, r_b)^2]^{1/2}$, the expectation values of $\mu(s)$ and $\sigma^2(s)$ over the test distribution are readily computed. Results are summarized in Figs. 1-3.

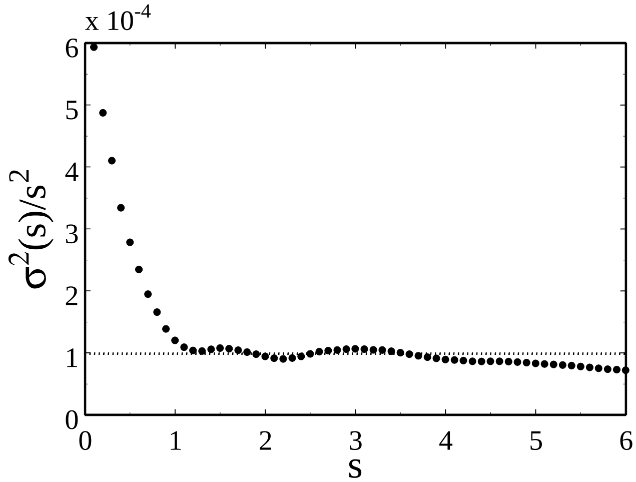

Figure 1: Plot of σ^2/s^2 as function of s.

Figure 1 shows plots of σ^2/s^2 versus the propagation distance s obtained from a self-consistent simulation of intense beam propagation through a sinusoidal periodic focusing channel. The choice of system parameters in Fig. 1 corresponding to $N_p = 1024$, $N_t = 512$, $K = 0.5$, and $\kappa_z(s) = [a_0 + a_1 \cos(2\pi s)]^2$, where $a_0 = a_1 = 0.648$. Due to small residual correlation in the initial distributions of test particles and background macroparticles, the value of σ^2/s^2 is large for $s \ll 1$. As the beam propagates, the residual correlation decays rapidly, and the value of σ^2/s^2 approaches a plateau for $s > 1$, where the diffusion coefficient is calculated to be $D = 1.0 \times 10^{-4}$ ($\overline{\xi} = 0.4$), as indicated by the dashed line. As the beam propagates further through the focusing channel, the plateau levels off because the test particles become widely spread about $A_{xi} = 1$.

The scaling law is verified by self-consistent simulations. Figure 2 shows a logarithmic plot of D versus K obtained from self-consistent simulations for beam propagation through the same periodic focusing channel as in Fig. 1. In Fig. 2, the number of background macroparticles is kept at a constant value of $N_p = 1024$. The dotted curve is from the self-consistent simulations, whereas the solid line is the analytical result given by $D = \alpha K^2$, where $\alpha = \overline{\xi}/N_p = 3.5\times10^{-4}$ ($\overline{\xi} = 0.35$). In Fig. 3, the diffusion coefficient D is plotted versus N_p, as obtained from self-consistent simulations of beam propagation through the same periodic focusing channel in Fig. 1 for a fixed value of $K = 0.5$. The dotted curve is from the self-consistent simulations, whereas the solid line is the analytical result given by $D = \beta K^2$, where $\beta = \overline{\xi} K^2 = 0.12$ ($\overline{\xi} = 0.48$). In comparison with Fig. 2, data fluctuations in Fig. 3 are larger because the initial distribution changes as N_p is varied. Nevertheless, it is evident in Fig. 2 and 3 that simulation results are in good agreement with the analytically predicted scaling law.

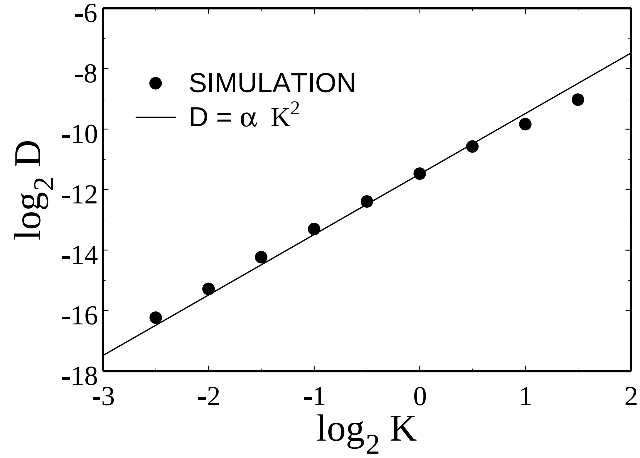

Figure 2: Log-log plot of D versus K.

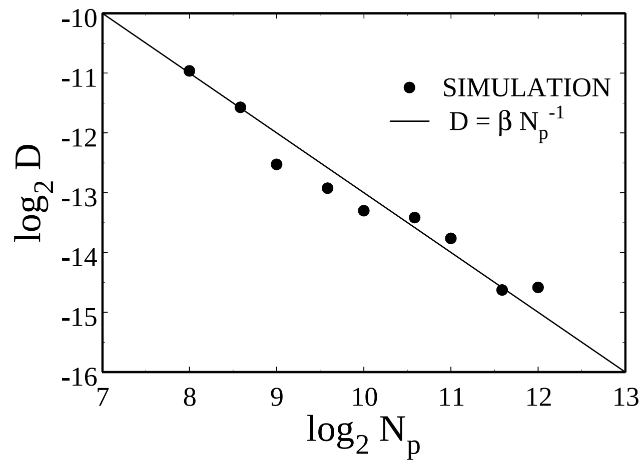

Figure 3: Log-log plot of D versus N_p.

To conclude, we have obtained a scaling law for edge emittance growth induced by discrete-particle effects in two dimensional self-consistent simulations of intense charged-particle beams in a periodic solenoidal focusing field. The scaling law may be applied to establish criteria for accurate simulation studies of the process of beam halo formation and beam losses.

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2 REFERENCES

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