THEORY OF LONGITUDINAL BEAM HALO IN RF LINACS: I. CORE/TEST-PARTICLE FORMULATION*

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Abstract

For intense beams, the analysis of tenuous halo components of the particle distribution that surround the main core of the beam can be challenging. So-called core/test-particle models in which a test-particle is evolved in the applied and space-charge forces of the beam core have been instrumental in understanding the structure and extent of transverse beam halo produced by resonant particle interactions with the oscillating space-charge forces of a mismatched beam core. Here we present a core/test-particle model developed for the analysis of longitudinal beam halo in intense, ion-beam rf linacs. Equations of motion are derived for a test-particle moving interior to, and exterior to, a uniform density ellipsoidal beam bunch. Coupled transverse-longitudinal mismatch modes of the ellipsoidal beam envelope are analyzed. Typical parameters suggest the possibility of a low-order resonant interaction between longitudinal particle oscillations and a low-frequency envelope mode. Properties of this resonance are analyzed in an accompanying paper by the authors in these proceedings[1].

1 INTRODUCTION

Ion linacs with high average current are being considered for applications such as the production of tritium and the transmutation of radioactive wastes. In such applications, beam halo can lead to a degradation of beam quality and/or particle losses resulting in activation of the accelerator. Therefore, the structure and control of halo components of the particle distribution is a critical issue. Recently, the understanding of transverse ($\perp$) beam halo has been advanced through analytic theory and numerical simulations[2−6]. In this and an accompanying article[1], we present theoretical and numerical work on longitudinal ($\|$) beam halo. Longitudinal halo particles are an issue of concern because such particles can have large-amplitude $\|$ oscillations about the synchronous particle, causing a degradation of $\|$ beam quality and possibly particle loss should the oscillation result in a loss of particle synchronisim with the rf fields. Moreover, the control of such halo losses could be challenging since the phase width of a beam bunch in the rf bucket cannot be made small in most intense-beam applications. In contrast, loss of $\perp$ halo particles can be mitigated, though with increased cost, through the use of large-aperture structures.

The core/test-particle model developed applies to a coasting ellipsoidal beam bunch that is continuously focused and has azimuthal symmetry. The space-charge of the bunch is assumed to remain uniformly distributed while undergoing coupled $\perp$ and $\|$ envelope mismatch oscillations, and details of the bunch velocity distribution are left unspecified. Self-field forces associated with the bunch are analytically calculated to obtain coupled equations of motion of a general test-particle undergoing both $\perp$ and $\|$ oscillations. These equations are employed to analyze $\|$ halo of on-axis particles (no $\perp$ motion) both with linear and nonlinear rf focusing, and to analyze effects of $\perp/\|$ coupling on halo particles (both $\perp$ and $\|$ motion).

To illustrate results, we employ the beam and accelerator parameters summarized in the Table. These parameters represent the 100 MeV and 1.2 GeV points of a conceptual coupled-cavity proton linac (normal conducting design) for the Accelerator Production of Tritium (APT) project[6]. In this design, an intense proton beam is accelerated from 100 MeV to 1.3 GeV over 1060 meters, and beam halo is an issue of concern.

Proton Energy, $\mathcal{E}_s$	GeV	0.1	1.0
Bunch Current, I	mA	200	200
Sync. Particle Phase, ϕ_s	degrees	-30	-30
rf Frequency, ν	MHz	700	700
Betatron "Freq.", $k_{\beta 0}$	rad/m	1.04	0.204
Synchrotron "Freq.", k_{s0}	rad/m	0.30	0.041
$\perp$ Norm. Emit., $\gamma_s\beta_s\epsilon_{x,rms}$	mm-mr	0.24	0.24
$\|$ Norm. Emit., $\gamma_s\beta_s\epsilon_{z,rms}$	mm-mr	0.58	0.62

Table 1: APT parameters in coupled cavity linac.

2 THEORETICAL MODEL

We consider an isolated ellipsoidal beam bunch composed of a single species of ion of charge q and mass m. The bunch is centered about a synchronous particle with phase $\phi = \phi_s$ relative to the peak of the synchronous space harmonic of the full rf wave. Acceleration is neglected, and the synchronous particle has $\|$ kinetic energy $\mathcal{E}_s$ =const. Beam focusing is provided transversely by a constant, linear applied field that represents the average effect of an alternating gradient focusing lattice, and longitudinally by a continuous sinusoidal wave that represents the average effect of the synchronous rf space harmonic. The bunch is taken to be azimuthally symmetric ($\partial/\partial\theta = 0$) and to have uniformly distributed charge-density $\bar{\rho}$ = const interior to a sharp ellipsoidal envelope specified by $(\mathbf{x}_\perp/r_\perp)^2 + (\Delta z/r_z)^2 = 1$, and zero charge-density exterior to the envelope. Here, $\mathbf{x}_\perp$ and Δz are the $\perp$ and $\|$ coordinates relative to the synchronous particle, and $r_\perp$ and r_z are the $\perp$ and $\|$ radii of the ellipsoidal beam enve-

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lope. The $\|$ coordinate Δz is related to the rf phase ϕ by $\Delta z = -(\beta_s \lambda/2\pi)\Delta\phi$, where $\Delta\phi \equiv \phi - \phi_s$, λ is the vacuum wavelength of the rf wave ($\lambda = c/\nu$, where c is the speed of light in vacuo and ν is the rf frequency), and β_s and $\gamma_s = 1/\sqrt{1-\beta_s^2}$ denote the usual synchronous particle relativistic factors. Denoting the time average current of the bunch over an rf period by I, the charge-density in the bunch is $\bar{\rho} = 3I\lambda/4\pi r_\perp^2 r_z c$.

The $\perp$ and $\|$ forces acting on a particle due to electrostatic and leading-order self-magnetic fields can be calculated in the absence of material boundaries as[7]

$$\begin{aligned} F_\perp &= \frac{q\bar{\rho}}{2\epsilon_0\gamma_s^2}\left[\frac{\alpha^2}{(\alpha^2+\chi)(1+\chi)^{1/2}} - F(\chi,\alpha)\right]\mathbf{x}_\perp, \\ \Delta F_z &= \frac{q\bar{\rho}}{\epsilon_0}F(\chi,\alpha)\Delta z. \end{aligned} \tag{1}$$

Here, ϵ_0 is the permittivity of free-space,

$$F(\chi,\alpha) \equiv \tag{2}$$

$$\begin{cases} \frac{\alpha^2}{1-\alpha^2}\left[\frac{1}{\sqrt{1-\alpha^2}}\tanh^{-1}\left(\frac{\sqrt{1-\alpha^2}}{\sqrt{1+\chi}}\right) - \frac{1}{\sqrt{1+\chi}}\right], & \alpha < 1, \\ \frac{1}{3(1+\chi)^{3/2}}, & \alpha = 1, \\ \frac{\alpha^2}{\alpha^2-1}\left[\frac{1}{\sqrt{1+\chi}} - \frac{1}{\sqrt{\alpha^2-1}}\tan^{-1}\left(\frac{\sqrt{\alpha^2-1}}{\sqrt{1+\chi}}\right)\right], & \alpha > 1, \end{cases}$$

$\alpha = r_\perp/\gamma_s r_z$ is the aspect ratio of the ellipsoidal beam as measured in the synchronous-particle frame, and χ is the positive root of the equation

$$\frac{\alpha^2(\mathbf{x}_\perp/r_z)^2}{\alpha^2+\chi} + \frac{(\Delta z/r_z)^2}{1+\chi} = 1 \tag{3}$$

for a particle exterior to the bunch envelope [i.e., $(\mathbf{x}_\perp/r_z)^2 + (\Delta z/r_z)^2 > 1$], and $\chi = 0$ for a particle interior to the bunch [i.e., $(\mathbf{x}_\perp/r_z)^2 + (\Delta z/r_z)^2 < 1$]. For the special cases of a particle in the beam, $\chi = 0$, and the self-field forces (1) reduce to the familiar linear expressions[8,9], and for a spherical bunch in the beam frame ($r_\perp = \gamma_s r_z$), the forces fall off with the required inverse-square form exterior to the bunch.

Within the paraxial approximation, the $\perp$ and $\|$ equations of motion of a single test-particle moving in the applied focusing fields and self-field defocusing forces (1) of the bunch can be expressed as

$$\begin{aligned} &\frac{d}{ds}\mathbf{x}_\perp = \mathbf{x}'_\perp, \qquad \frac{d}{ds}\Delta\phi = -\frac{2\pi}{\gamma_s^3\beta_s^3\lambda}\frac{\Delta\mathcal{E}}{mc^2}, \\ &\frac{d}{ds}\mathbf{x}'_\perp = -k_{\beta 0}^2\mathbf{x}_\perp + \frac{K_{3D}}{2r_\perp^2 r_z}\left[\frac{\alpha^2}{(\alpha^2+\chi)(1+\chi)^{1/2}} - F(\chi,\alpha)\right]\mathbf{x}_\perp, \\ &\frac{d}{ds}\left(\frac{\Delta\mathcal{E}}{mc^2}\right) = \frac{qE_0}{mc^2}\left[\cos(\Delta\phi+\phi_s) - \cos\phi_s\right] \\ &\qquad - \frac{\beta_s^3\gamma_s^3}{2\pi}\frac{\lambda K_{3D}}{r_\perp^2 r_z}F(\chi,\alpha)\Delta\phi. \end{aligned} \tag{4}$$

Here, s is the axial distance the beam has propagated, $\mathbf{x}_\perp$ and $\mathbf{x}'_\perp$ are the $\perp$ coordinate and convergence angle of the particle, $\Delta\phi$ and $\Delta\mathcal{E} = \mathcal{E} - \mathcal{E}_s$ are the $\|$ particle phase and kinetic energy relative to the synchronous particle, $k_{\beta 0}$ =const is the undepressed ($I = 0$) betatron wavenumber of $\perp$ particle oscillations, E_0 ($E_0 \to E_0 T$ for a standing-wave structure, with T the transit-time factor) is the peak, on-axis ($\mathbf{x}_\perp = 0$) field value of rf wave, and $K_{3D} \equiv 3qI\lambda/4\pi\epsilon_0\gamma_s^3\beta_s^2 mc^3$ is a three-dimensional space-charge parameter of dimension length. In general, note that the $\perp$ and $\|$ Eqs. (4) are coupled due to the coordinate dependence in χ. However, for a particle moving within the bunch (i.e., $\chi = 0$) or along the axes of symmetry of the bunch (i.e., $\mathbf{x}_\perp = 0$ or $\Delta\phi = 0 = \Delta z$), the equations are uncoupled. Finally, for small $\|$ particle excursions from the synchronous particle with $2\pi|\Delta z|/\beta_s\lambda \ll 1$, the $\|$ rf focusing force becomes linear, i.e.,

$$\frac{qE_0}{\beta_s^2\gamma_s^3 mc^2}\left[\cos(\Delta\phi+\phi_s) - \cos\phi_s\right] \longrightarrow k_{s0}^2\Delta z, \tag{5}$$

where $k_{s0} \equiv \sqrt{2\pi qE_0\sin(-\phi_s)/\gamma_s^3\beta_s^3\lambda mc^2}$ is the wavenumber of undepressed "synchrotron" oscillations about the synchronous particle. In the presence of space-charge ($I \neq 0$) and within the bunch, it follows from Eqs. (4) that the characteristic $\perp$ and $\|$ spatial frequencies of undepressed particle oscillations $k_{\beta 0}$ and k_{s0} are depressed to

$$\begin{aligned} k_\beta^2 &= k_{\beta 0}^2 - \frac{K_{3D}}{2r_\perp^2 r_z}[1 - f(\alpha)], \\ k_s^2 &= k_{s0}^2 - \frac{K_{3D}}{r_\perp^2 r_z}f(\alpha), \end{aligned} \tag{6}$$

where $f(\alpha) \equiv F(\chi = 0, \alpha)$ is a beam aspect ratio form factor [corresponding to f in Ref. 8].

3 ENVELOPE EQUATIONS

To consistently determine the ellipsoidal bunch radii $r_\perp$ and r_z in terms of beam and accelerator parameters, it is necessary to derive so-called "envelope equations" for the evolution of $r_\perp$ and r_z. For a uniform density distribution, $r_\perp^2 = (5/2)\langle r^2\rangle$ and $r_z^2 = 5\langle\Delta z^2\rangle$, where $\langle\rangle$ indicates an average over the entire 3D distribution function. By taking derivatives of these relations and using the equations of motion (4), the envelope equations

$$\begin{aligned} \frac{d^2 r_\perp}{ds^2} &= -k_{\beta 0}^2 r_\perp + \frac{K_{3D}[1-f(\alpha)]}{2r_\perp r_z} + \frac{\epsilon_x^2}{r_\perp^3} \\ \frac{d^2 r_z}{ds^2} &= -f_{rf}(\zeta)\, k_{s0}^2 r_z + \frac{K_{3D}f(\alpha)}{r_\perp^2} + \frac{\epsilon_z^2}{r_z^3} \end{aligned} \tag{7}$$

are obtained[8,9]. Here, $f(\alpha) = F(\chi = 0, \alpha)$, $\epsilon_x^2 \equiv 25[\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2] = 25\epsilon_{x,rms}^2$ and $\epsilon_z^2 \equiv 25[\langle\Delta z^2\rangle\langle\Delta z'^2\rangle - \langle\Delta z\Delta z'\rangle^2] = 25\epsilon_{z,rms}^2$ are the squares of the unnormalized $\perp$ and $\|$ 3D bunch emittances, and

$$f_{rf}(\zeta) = \frac{15}{\zeta^5}[(3-\zeta^2)\sin\zeta - 3\zeta\cos\zeta] \tag{8}$$

with $\zeta \equiv (2\pi/\beta_s\lambda)r_z$ is a nonlinear rf focusing factor.

Note that Eqs. (7) differs from those in Refs. 8 and 9 by the inclusion of a multiplicative factor $f_{rf}(\zeta)$, which arises when the external rf force is included in an average over the $\|$ coordinate. This factor approaches unity as $\zeta \to 0$, corresponding to small $\|$ beam extent r_z relative to the rf wavelength λ. Note that Eqs. (7) are not self-consistent in that the distribution function is *assumed* to evolve such that its density remains uniform within an ellipsoid, whereas in general, such evolution does not occur. It is hoped that, since the extent of the distribution is not constrained (though the form of the distribution is), the evolution of these envelopes is close to the actual evolution of rms measures of the radii, thereby providing halo particles with approximately correct space-charge impulses.

4 LINEAR ENVELOPE MODES

Denote stationary ($d/ds = 0$) equilibrium (subscript 0) solutions to the envelope equations (7) by $r_{\perp 0}$ and r_{z0}, and assume coupled, sinusoidally varying perturbations about the equilibrium radii of the form $r_\perp = r_{\perp 0} + \delta r_\perp \mathrm{e}^{iks}$ and $r_z = r_{z0} + \delta r_z \mathrm{e}^{iks}$ with constant amplitudes $\delta r_\perp$ and δr_z. For $|\delta r_{\perp,z}|/r_{\perp 0,z0} \ll 1$ these perturbations can be expanded to leading order in Eq. (7) to obtain the dispersion relation

$$k^4 - (K_{11} + K_{22})k^2 + K_{11}K_{22} - K_{12}K_{21} = 0 \qquad (9)$$

for the mode wavenumber or spatial frequency k. Here,

$$\begin{aligned}
K_{11} &= 4k_{\beta 0}^2 - \frac{K_{3D}}{r_{\perp 0}^2 r_{z0}}\left[1 - f(\alpha_0) - \frac{\alpha_0}{2}\frac{df(\alpha_0)}{d\alpha_0}\right], \\
K_{22} &= 4k_{s0}^2\left[f_{rf}(\zeta_0) + \frac{\zeta_0}{4}\frac{df_{rf}(\zeta_0)}{d\zeta_0}\right] \\
&\quad - \frac{3K_{3D}}{r_{\perp 0}^2 r_{z0}}\left[f(\alpha_0) - \frac{\alpha_0}{3}\frac{df(\alpha_0)}{d\alpha_0}\right], \qquad (10) \\
K_{12} &= \frac{K_{3D}}{2r_{\perp 0}^2 r_{z0}}\left[1 - f(\alpha_0) - \alpha_0\frac{df(\alpha_0)}{d\alpha_0}\right], \\
K_{21} &= \frac{K_{3D}}{r_{\perp 0}^2 r_{z0}}\left[2f(\alpha_0) - \alpha_0\frac{df(\alpha_0)}{d\alpha_0}\right],
\end{aligned}$$

are equilibrium parameters. Since Eq. (9) is quadratic in k^2, two modes are present, a high frequency (HFM, $k \equiv k_H$) and low frequency mode (LFM, $k \equiv k_L$), each with a relative amplitude of $\perp$ to $\parallel$ oscillation given by

$$R \equiv \frac{\delta r_\perp / r_{\perp 0}}{\delta r_z / r_{z0}} = \frac{K_{12}}{k^2 - K_{11}}. \qquad (11)$$

In the Figure, equilibrium and mode properties are presented as a function of bunch current I assuming fixed $\perp$ and $\parallel$ beam emittances ϵ_x and ϵ_z. Parameters correspond to the 100 MeV linac presented in the Table with I varied about the nominal 200 mA value.

5 CONCLUSIONS

From Figure a and c, it is apparent that for beams in which the $\perp$ / $\parallel$ applied focusing and equilibrium radii ratios $k_{\beta 0}/k_{s0}$ and $r_{z0}/r_{\perp 0}$ are of order a few, the frequencies of the HFM (k_H) and LFM (k_L) are well separated. (However, for the more recent superconducting design of APT, the ratio $r_{z0}/r_{\perp 0}$ is closer to unity[10] implying less separation of the oscillation frequencies.) From Figure c, note that the LFM is antisymmetric, with $\perp$ envelope excursions much smaller than $\parallel$ excursions, and 180 degrees out of phase. In contrast, the HFM is symmetric, with $\parallel$ excursions much smaller than $\perp$ excursions and in-phase. The frequency of the LFM falls between twice the $\parallel$ particle frequency within the beam ($\sim 2k_s$), and twice the undepressed frequency ($\sim 2k_{s0}$, i.e. twice the $\parallel$ particle frequency at large $|\Delta z|$ in the linear rf focusing approximation). Particles outside the beam envelope longitudinally will have an oscillation frequency between k_s and k_{s0}, suggesting a possible low-order (2:1) resonant interaction between $\parallel$ particle and envelope motion. This situation is analogous to the transverse case[2–5] and has analogous implications for longitudinal beam halo. This longitudinal resonance is analyzed in an accompanying paper[1].

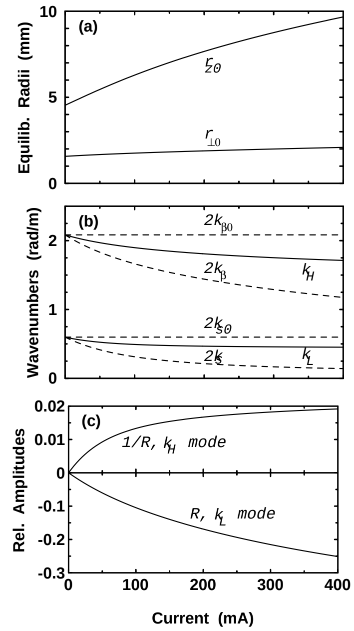

Figure 1: Ellipsoidal beam equilibrium and mode properties as a function of current. **a.** Equilibrium beam radii $r_{\perp 0}$ and r_{z0}. **b.** Wavenumbers for high (k_H) and low (k_L) frequency envelope modes (solid curves), and two times the undepressed and depressed frequencies of $\perp$ ($k_{\beta 0}$, k_β) and $\parallel$ (k_{s0}, k_s) particle oscillations (dashed curves). **c.** Ratio R of $\perp$ to $\parallel$ mode amplitude for the high and low frequency envelope modes.

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7 REFERENCES

[1] S.M. Lund and J.J. Barnard, "Theory of Longitudinal Beam Halo in RF Linacs: II. Envelope/Particle Resonances," these proceedings.

[2] J.S. O'Connell, T.P. Wangler, K.R. Krandall, Proc. 1993 Part. Accel. Conf., Washington DC, p. 3657.

[3] R.A. Jameson, loc. sit. 2, p. 3926.

[4] R.L. Gluckstern, Phys. Rev. Letters, **73**, 1247, (1994).

[5] R.D. Ryne, T.P. Wangler, Proc. Accel. Driven Transmutation Tech. Conf., Las Vegas, NV, July 1994.

[6] J. Stovall et al., APT Design Review Presentation, Los Alamos National Laboratory, Oct., 1996.

[7] J.J. Barnard and S.M. Lund, "Theory of Longitudinal Beam Halo in High Current Ion RF Linacs," in prep.

[8] T.P. Wangler, "Introduction to Linear Accelerators," Los Alamos Report, LA-UR-94-125 (1994).

[9] M. Reiser, "Theory and Design of Charged Particle Beams," (Wiley & Sons, New York, 1994).

[10] T.P. Wangler, private communication (1997).