DISPERSION IN THE PRESENCE OF STRONG TRANSVERSE WAKEFIELDS*

Ralph W. Assmann and Alex Chao
Stanford Linear Accelerator center, Stanford University, Stanford, CA 94309

Abstract

To minimize emittance growth in a long linac, it is necessary to control the wakefields by correcting the beam orbit excursions. In addition, the particle energy is made to vary along the length of the bunch to introduce a damping, known as the BNS damping, to the beam break-up effect. In this paper, we use a two-particle model to examine the relative magnitudes of the various orbit and dispersion functions involved. The results are applied to calculate the effect of a closed orbit bump and a misaligned structure. It is shown that wake-induced dispersion is an important contribution to the beam dynamics in long linacs with strong wakefields like SLC.

1 WAKE INDUCED DISPERSION (TWO-PARTICLE MODEL)

Consider a linac with uniform betatron focusing and no acceleration. Introduce an orbit kick θ at $s = 0$. The betatron equation of motion for a particle with relative energy error δ is

$$x''(s) + \frac{k_\beta^2}{1+\delta} x(s) = \frac{\theta}{1+\delta}\delta(s), \tag{1}$$

with the solution

$$x(s) = \frac{\theta}{k_\beta\sqrt{1+\delta}} \sin(\frac{k_\beta s}{\sqrt{1+\delta}}). \tag{2}$$

When $k_\beta s\delta \ll 1$, one may expand (2) in δ, i.e.

$$x(s) = x_0(s) + \eta(s)\delta + \mathcal{O}(\delta^2), \tag{3}$$

with

$$x_0(s) = \frac{\theta}{k_\beta}\sin k_\beta s$$

and the dispersion function $\eta(s) = -\frac{\theta}{2}(s\cos k_\beta s + \frac{1}{k_\beta}\sin k_\beta s)$. When $k_\beta s \gg 1$, the dispersion effect can clearly be important. When $k_\beta s\delta \ll 1$ is not satisfied, we will have to use Eq.(2) instead of Eq.(3).

We next consider a two-particle model for the kicked beam. The motion of the leading macroparticle (considered to be on-momentum) of the beam is given by $x = x_0(s)$. Let $N/2$ be the number of electrons in the leading and the trailing macroparticles, γ be the design energy Lorentz factor, W_1 be the wake function per cavity period, and L be the cavity period length. Let $y(s)$ designate the orbit deviation of the trailing macroparticle. We have

$$\begin{aligned} y''(s) + \frac{k_\beta^2}{1+\delta}y(s) &= \frac{\theta}{1+\delta}\delta(s) \\ &- \frac{Nr_0W_1}{2\gamma L(1+\delta)}\frac{\theta}{k_\beta}\sin k_\beta s, \end{aligned} \tag{4}$$

where r_0 is the classical electron radius. In Eq.(5), we have assumed the leading and the trailing macroparticles have the same design betatron frequency. This is the case when there is no BNS damping. The case with BNS damping is to be treated later.

The solution to Eq.(5) is

$$\begin{aligned} y(s) = x(s) - \frac{\Upsilon\theta}{k_\beta L_0}\frac{2}{k_\beta\delta}\cdot \\ \left(\sin k_\beta s - \sqrt{1+\delta}\sin\frac{k_\beta s}{\sqrt{1+\delta}}\right) \end{aligned} \tag{5}$$

where we have defined a dimensionless parameter

$$\Upsilon = -\frac{Nr_0W_1L_0}{4\gamma Lk_\beta}. \tag{6}$$

The first term on the right hand side of Eq.(6) is the direct response of the trailing macroparticle to the orbital kick and is the same as Eq.(2). The second term is the driven response to the wakefield.

When $k_\beta s\delta \ll 1$, we can expand (6) in δ to obtain

$$\begin{aligned} y(s) &= y_0(s) + [\eta(s) + \xi(s)]\delta + \mathcal{O}(\delta^2) \\ y_0(s) &= x_0(s) - \frac{\Upsilon\theta}{k_\beta L_0}(s\cos k_\beta s - \frac{1}{k_\beta}\sin k_\beta s) \\ \xi(s) &= -\frac{\Upsilon\theta}{4k_\beta L_0}\cdot \\ &\left(k_\beta s^2\sin k_\beta s - s\cos k_\beta s + \frac{1}{k_\beta}\sin k_\beta s\right) \end{aligned} \tag{7}$$

where $y_0(s)$ is the usual beam break-up response to wake fields, and $\xi(s)$ is the wake-induced dispersion function.

Note that $\xi(s)$ is doubly resonantly driven as evidenced by its containing a term proportional to s^2. When Υ and $k_\beta L_0$ are both $\gg 1$, the ratio among the four quantities is

$$\xi\delta \ : \ y_0 \ : \ \eta\delta \ : \ x_0 \ \approx \ \frac{\Upsilon k_\beta L_0\delta}{4} \ : \ \Upsilon \ : \ \frac{k_\beta L_0\delta}{2} \ : \ 1. \tag{8}$$

Comparing ξ with η near the end of linac, the magnitude of ξ is larger by a factor of $\Upsilon/2$. This indicates the wake-induced dispersion may be an important consideration in a long linac such as the SLC or the NLC.

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In the above analysis, acceleration has been ignored. When acceleration is taken into account, we need to replace the expression (7) by

$$\Upsilon = -\frac{Nr_0 W_1 L_0}{4\gamma_f L k_\beta} \ln \frac{\gamma_f}{\gamma_i}, \tag{9}$$

where $\gamma_{i,f}$ refer to the initial and final beam energies of the linac.

For the SLC, if we take $N = 5 \times 10^{10}$, $W_1 = -0.7$ cm^{-2}, $L_0 = 3$ km, $L = 3.5$ cm, and $k_\beta = 0.06$ m^{-1}, and let the beam be accelerated from 1 GeV to 50 GeV, we find $\Upsilon = 14$. If we further take $\delta = 0.5\%$, we find $\xi\delta : y_0 : \eta\delta : x_0 \approx$ 3.1 : 14 : 0.45 : 1.

One may ask what happens when a BNS damping is imposed. In this case, the leading macroparticle sees a focusing gradient k_β, therefore $x(s), x_0(s)$ and $\eta(s)$ remain given by Eqs.(2-4). However, the trailing particles see a stronger focusing with

$$y''(s) + \frac{(k_\beta + \Delta k_\beta)^2}{1+\delta} y(s) = \frac{\theta}{1+\delta}\delta(s) - \frac{Nr_0 W_1}{2\gamma L(1+\delta)} \frac{\theta}{k_\beta} \sin k_\beta s \tag{10}$$

which has the solution

$$y(s) = \frac{\theta}{(k_\beta + \Delta k_\beta)\sqrt{1+\delta}} \sin(\frac{k_\beta s + \Delta k_\beta s}{\sqrt{1+\delta}}) - \frac{2\Upsilon\theta}{k_\beta L_0} \frac{1}{1+\delta - (1+\frac{\Delta k_\beta}{k_\beta})^2} \left[\frac{1}{k_\beta}\sin k_\beta s - \frac{\sqrt{1+\delta}}{k_\beta + \Delta k_\beta} \sin(\frac{k_\beta s + \Delta k_\beta s}{\sqrt{1+\delta}})\right]. \tag{11}$$

The BNS condition is to choose Δk_β such that

$$(1 + \frac{\Delta k_\beta}{k_\beta})^2 = 1 + \frac{2\Upsilon}{k_\beta L_0}. \tag{12}$$

When (13) is satisfied, the orbit of a trailing particle whose $\delta = 0$ is identical to that of the leading macroparticle, i.e. $y(s, \delta = 0) = x_0(s)$, thus minimizing the beam emittance growth due to wake fields. The question now is what happens to the wake-induced dispersion effect. With the BNS condition (13), Eq.(12) reads

$$y(s) = x_0(s) + \frac{\theta\delta}{k_\beta(\delta - \frac{2\Upsilon}{k_\beta L_0})} \left[\sqrt{\frac{1+\frac{2\Upsilon}{k_\beta L_0}}{1+\delta}} \sin\left(k_\beta s \sqrt{\frac{1+\frac{2\Upsilon}{k_\beta L_0}}{1+\delta}}\right) - \sin k_\beta s\right]. \tag{13}$$

One sees a resonance response when $\delta = \frac{2\Upsilon}{k_\beta L_0}$. This is because then the tail particle has a betatron focusing strength $\frac{k_\beta + \Delta k_\beta}{\sqrt{1+\delta}} = k_\beta$.

Typically we have $\frac{\Upsilon}{k_\beta L_0} \ll 1$ (and thus $\Delta k_\beta \ll k_\beta$) and $\delta \ll 1$. If we further have the condition $k_\beta s \delta \ll 1$ and $\delta \ll \frac{2\Upsilon}{k_\beta L_0}$, then we can write

$$y(s) \approx x_0(s) + \xi(s)\delta, \tag{14}$$

where

$$\xi = -\frac{\theta L_0}{2\Upsilon}[\sin(k_\beta s + \frac{\Upsilon}{L_0}s) - \sin k_\beta s]. \tag{15}$$

This is a very small dispersion. We have, instead of Eq.(9),

$$\xi\delta \,:\, y \,:\, \eta\delta \,:\, x = \frac{k_\beta L_0 \delta}{2\Upsilon} \,:\, 1 \,:\, \frac{k_\beta L_0 \delta}{2} \,:\, 1. \tag{16}$$

Note that $y(s)$ in (15) does not contain a term $\eta(s)\delta$ as Eq.(8) did. Note also that inspite of BNS, ξ is not identical to η, although y is made identical to x by the BNS condition. There is therefore a dispersive mismatch between the head and the tail of the bunch due to a mismatch between $\xi\delta$ and $\eta\delta$. The increase in the beam emittance is

$$\Delta\epsilon = \frac{k_\beta}{2}(\frac{1}{2}\theta L_0 \sigma_\delta)^2 \tag{17}$$

In order for this effect to be negligible, we need the condition, even when the BNS condition is perfectly satisfied,

$$\theta L_0 \sigma_\delta \ll \sigma_\beta, \tag{18}$$

where σ_β is the betatron beam size.

2 CLOSED π BUMPS

So far, we have considered the case of an uncorrected betatron oscillation, induced by a single kick θ. We now study a closed π-bump that is implemented with two kicks θ that are located at $s = 0$ and $s = \pi/k_\beta$. The orbit and dispersion functions for the second kick are obtained by substituting s with $s - \pi/k_\beta$. For example, $x_0(s) \to x_0'(s - \pi/k_\beta)$.

The orbit function $\tilde{x}_0$ downstream of the π-bump is just the sum of x_0 and x_0'. With Eq.(3) we obtain immediately $\tilde{x}_0 = 0$, indicating that the bump is closed. Doing the same excercise for the tail orbit and the dispersion functions we obtain from Eqs.(4) and (8):

$$\begin{aligned}
\tilde{\eta}(s) &= \eta(s) + \eta(s - \pi/k_\beta) = -\frac{\theta\pi}{2k_\beta}\cos(k_\beta s) \\
\tilde{y_0}(s) &= -\frac{\pi\Upsilon\theta}{k_\beta^2 L_0}\cos k_\beta s \\
\tilde{\xi}(s) &= -\frac{\pi\Upsilon\theta}{4k_\beta^2 L_0}\left[(2k_\beta s - \pi)\sin k_\beta s - \cos k_\beta s\right].
\end{aligned} \tag{19}$$

We obtain the well known result that a closed π-bump is not closed for the dispersion η, and generates a dispersion oscillation with constant amplitude. The π-bump is also not closed for the tail orbit. If the BNS condition is satisfied the bump is closed for both head and tail particle. The results are important because the orbit after steering can be described by a superposition of π-bumps (90 degree lattice). The wake kick during the orbit bump, together with the energy offset of the tail particle, gives rise to a wake-induced dispersion which increases linearly with s.

3 DISPERSION FROM A MISALIGNED STRUCTURE

The kick θ above applied both to head and tail particle, as from quadrupole offsets or dipole correctors. In the case of a structure offset, the head induces a dipole wakefield that deflects the tail particle by $\hat{\theta}$. The head orbit is not disturbed and we can write:

$$x_0(s) = 0 \quad , \quad y_0(s) = \frac{\hat{\theta}}{k_\beta} \sin k_\beta s. \tag{20}$$

Orbit correction (without BPM errors) requires that the sum of head and tail trajectory is zero ($x_0(s)+y_0(s)=0$). This is achieved by applying a kick $\theta = -\hat{\theta}/2$ to both particles. Assuming BNS and $\delta = 0$ we get:

$$x_0(s) = -y_0(s) = \frac{\theta/2}{k_\beta} \sin k_\beta s. \tag{21}$$

The centroid trajectory is zero as required, but head and tail particle perform uncorrected betatron oscillations. The centroid dispersion η_{tot} is:

$$\begin{aligned} \eta_{tot} &= -\left(\frac{\hat{\theta}}{2} + \frac{\theta}{2}\right)(s \cos k_\beta s + \frac{1}{k_\beta} \sin k_\beta s) \\ &- \frac{\theta L_0}{2\Upsilon}[\sin(k_\beta s + \frac{\Upsilon}{L_0} s) - \sin k_\beta s]. \end{aligned} \tag{22}$$

Because the tail dispersion ξ that is induced by the "corrector" kick θ is small (see Eqs.(17)), we can neglect it here. With $\theta = -\hat{\theta}/2$ we obtain:

$$\eta_{tot} \approx -\frac{\hat{\theta}}{4}(s \cos k_\beta s + \frac{1}{k_\beta} \sin k_\beta s) \tag{23}$$

The centroid dispersion η_{tot} that is generated by a misaligned structure grows resonantly with s and therefore becomes large for long linacs.

The orbit due to a misaligned quadrupole and after trajectory correction can be described by a π-bump through the quadrupole center (90 degree lattice). The resulting dispersion downstream of the closed bump was given in Eq.(20). Comparing this to Eq.(24), we see that the dispersion generated from a misaligned structure can become larger than the dispersion from a misaligned quadrupole after orbit correction. The ratio between them is of the order of $(\hat{\theta}/\theta) \cdot (k_\beta L_0/2\pi)$ at the end of the linac. Due to its resonant growth in s, dispersion from a misaligned structure becomes larger than the one from a misaligned quadrupole though $\hat{\theta} < \theta$ for the same misalignment. The importance of wake-induced dispersion was indeed shown in SLC simulations done with the computer program LIAR [1]. Figure 1 shows that wakefield generated dispersion in the SLC becomes larger than quadrupole generated dispersion for large bunch charges.

4 CONCLUSION

The orbit and dispersion functions in the presence of wakefields have been calculated. It was shown that the dispersion effect of an orbit kick is made much worse by the presence of the wakefields in the absence of BNS damping. The effect of this large wake-induced dispersion is found to be suppressed but not removed when BNS condition is introduced. The results were used to evaluate the effects of a closed orbit π-bump and a misaligned structure. The dispersion generated from a structure offset was shown to grow resonantly with s *after* trajectory correction. Therefore, wakefield generated dispersion can become much larger than the dispersion from misaligned quadrupoles. Simulations for the SLC linac confirmed this behavior [2].

As centroid dispersion with strong wakefields is mainly generated by structure offsets, a dispersion measurement can be used to determine the structure errors. We can envision new and improved algorithms to optimize emittance in linacs with strong wakefields. For example, emittance might be optimized by empirically adjusting structure movers so as to minimize the measured centroid dispersion.

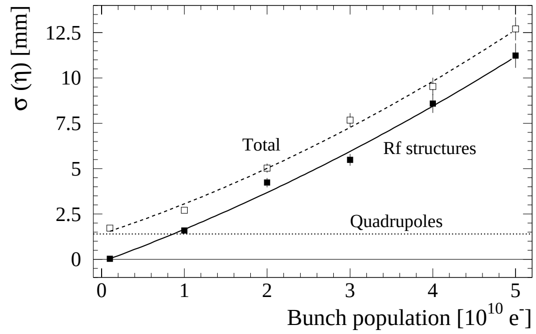

Figure 1: Simulated RMS dispersion as a function of bunch population at the end of the SLC linac and after orbit correction. The dotted line shows the dispersion without wakefields. The points indicate simulation results with wakefields. The filled points show a case where only the RF structures were misaligned. Finally, the open points show the total dispersion, if quadrupole and BPM errors are included. It is seen that centroid dispersion is mainly generated by structure offsets for the SLC at high bunch current. Even with a flat centroid trajectory (filled points), the centroid dispersion can become large.

5 ACKNOWLEDGEMENTS

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6 REFERENCES

[1] R. Assmann et al, "LIAR - A Computer Program for the Modeling and Simulation of High Performance Linacs". SLAC/AP-103.

[2] R. Assmann et al., "Dispersion measurements and studies for the SLC linac". To be published.