

# CONSIDERATION OF DC SPACE CHARGE EFFECTS IN THE MODAL ANALYSIS OF APERTURE-COUPLED CAVITIES

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*Abstract*

The modal analysis of aperture-coupled cavities is frequently applied to the field theoretical analysis of high-power tube cavities and accelerating structures in high energy physics. Besides the rf electromagnetic field, dc space charge effects may also be important in such applications. Because there is no magnetic field related with dc space charges a straightforward application of the modal expansion method, which makes use of matching the aperture tangential magnetic field, to the investigation of dc space charge fields is not possible. In this contribution it is therefore demonstrated that the computation of the dc spectral component of the electric field due to space charges can also be included in the modal analysis of aperture-coupled cavities if instead of the aperture tangential magnetic field the corresponding normal electric field is matched. It is shown that even in this case the solenoidal electric eigenfunctions of the cavity have to be taken into account because of the waveguide coupling. Furthermore, the validity of the method is checked by applying it to a field problem for which a solution is already well-known from another analysis.

## 1 INTRODUCTION

In many high-power tubes, e.g. klystrons, travelling wave tubes and gyrotrons, the beam-field interaction takes place in one or more aperture-coupled cavities. In the analysis of such tubes space charge effects are often not taken into account [1]–[3] although it is well-known that this effect cannot always be neglected. This is especially the case if high charge densities are encountered, as e.g. in the output cavity of a klystron. In [4] a modal analysis of aperture-coupled cavities has been presented which takes space charge effects into account: By the application of the equivalence principle [5], the coupling-apertures are short-circuited and the nonvanishing tangential electric field there is replaced by two surface magnetic currents at both sides of the short circuit, which are equal in magnitude and opposite in direction. The electromagnetic field inside the cavity is then expanded with respect to the solenoidal *and* irrotational eigenfunctions of the corresponding completely shielded cavity including the effect of space charges in the analysis.

The method presented in [4] makes use of matching the aperture tangential magnetic field. Consequently this analysis cannot be used for the computation of the electric field at  $\omega = 0$  which is just the electrostatic field generated by the space charges. In this contribution it is shown how the modal analysis is to be modified so that it can also be ap-

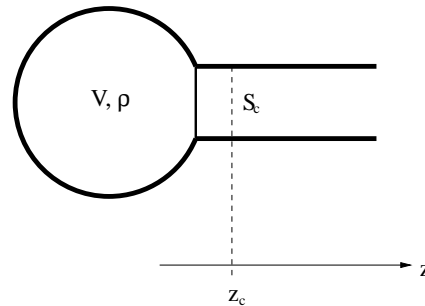


Figure 1: Cavity which is coupled to a single waveguide.

plied to the analysis of the electrostatic field. Nevertheless it turns out that even in this case solenoidal electric fields are excited due to the magnetic surface currents at the short-circuited coupling apertures.

## 2 BASIC FORMULATION

For the sake of simplicity let us consider a cavity with only one coupling aperture as shown in Fig. 1. The extension of the method to more than one aperture is straightforward. In [4] the electromagnetic field inside the cavity is expanded in terms of the solenoidal ( $e_n, h_n$ ) *and* irrotational ( $f_n, g_n$ ) eigenfunctions of the corresponding completely shielded resonator:

$$\mathbf{E} = \sum_n a_n \mathbf{e}_n + \sum_n b_n \mathbf{f}_n \quad , \quad (1)$$

$$\mathbf{H} = \sum_n c_n \mathbf{h}_n + \sum_n d_n \mathbf{g}_n \quad (2)$$

The expansion coefficients  $b_n$  corresponding to the irrotational electric eigenfunctions are given by

$$\frac{b_n}{\bar{k}_{0n}} = -\frac{Z_0}{jk_0} \int_V \mathbf{J} \cdot \mathbf{f}_n^* dV \quad , \quad (3)$$

where  $Z_0, k_0, \bar{k}_{0n}$  and  $\mathbf{J}$  denote the intrinsic impedance of free space, the vacuum wavenumber which is proportional to the operating frequency, the eigenvalue corresponding to  $\mathbf{f}_n$  and the current density, respectively. Keeping in mind that the irrotational electric eigenfunctions are related to potentials  $\varphi_n$  by

$$\mathbf{f}_n = \nabla \varphi_n \quad \text{with} \quad \varphi_n = 0 \quad \text{on} \quad S \quad , \quad (4)$$

where  $S$  is the surface of the corresponding completely shielded cavity, and by making use of the continuity equation

$$\nabla \cdot \mathbf{J} = -jk_0 c_0 \varrho \quad , \quad (5)$$

where  $c_0$  and  $\varrho$  are the free space velocity of light and the charge density, respectively, (3) can be reformulated:

$$\frac{b_n}{k_{0n}} = -Z_0 c_0 \int_V \varrho \varphi_n^* dV \quad (6)$$

Thus the coefficient  $b_n$  is simply given by the expansion of the charge density  $\varrho$  with respect to the potential  $\varphi_n$ . On the other hand, it can be shown that the irrotational magnetic eigenfunctions are not excited.

Although we consider an electrostatic problem the expansion coefficients  $a_n$  corresponding to the solenoidal electric cavity eigenfunctions  $e_n$  do not vanish due to the coupling apertures. From the modal analysis we obtain

$$a_n = j \int_{S_c} (\mathbf{E} \times \hat{\mathbf{k}}) \cdot \mathbf{h}_n^* dS \quad (7)$$

for  $\omega = 0$  where  $S_c$  and  $\hat{\mathbf{k}}$  are the coupling aperture and the unit vector in axial direction, respectively. Note that (7) still holds in the presence of a dc magnetic field due to a stationary current inside the cavity because electric and magnetic fields are decoupled for  $\omega = 0$ . For the field inside the waveguide only TM-modes have to be taken into account since the dc electric field of TE-modes vanishes. On the other hand, TM-modes have no magnetic field for  $\omega = 0$  so that the waveguide field reads

$$\mathbf{E}_t = \sum_{\mu} A_{\mu} \nabla_t e_{z\mu} \exp[-k_{\mu}(z - z_c)] \quad , \quad (8)$$

$$E_z = -\sum_{\mu} A_{\mu} k_{\mu} e_{z\mu} \exp[-k_{\mu}(z - z_c)] \quad . \quad (9)$$

In (8) and (9),  $\nabla_t$ ,  $e_{z\mu}$ ,  $k_{\mu}$  and  $z_c$  are the transverse component of the nabla-operator, the axial electric field characterizing the  $\mu$ th TM-mode, the corresponding cutoff wavenumber and the axial coordinate of the coupling aperture, respectively. Inserting (8) into (7), which guarantees the continuity of the aperture tangential electric field, one arrives at

$$a_n = -j \sum_{\mu} A_{\mu} \int_{S_c} (\nabla_t e_{z\mu} \times \mathbf{h}_n^*) \cdot \hat{\mathbf{k}} dS \quad . \quad (10)$$

By making use of Maxwell's equation

$$\nabla \times \mathbf{h}_n = jk_{0n} e_n \quad (11)$$

for the normalized cavity eigenfunctions  $e_n$  and  $\mathbf{h}_n$ , where  $k_{0n}$  is the corresponding resonant wavenumber, we find after some algebraic manipulations that the cavity expansion coefficients  $a_n$  are related with the waveguide mode amplitudes  $A_{\mu}$  by a linear transformation:

$$a_n = \sum_{\mu} K_{n,\mu} A_{\mu} \quad \text{with} \quad (12)$$

$$K_{n,\mu} = k_{0n} \int_{S_c} e_{z\mu} e_n^* \cdot \hat{\mathbf{k}} dS \quad (13)$$

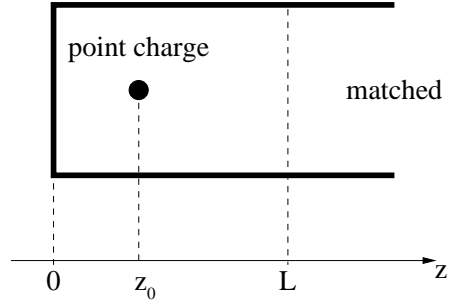


Figure 2: Short-circuited waveguide which contains a point charge.

Note that this transformation is purely real because the electric field is in phase throughout the whole structure for an electrostatic problem. Instead of matching the aperture tangential magnetic field, we exploit the continuity of the axial electric field which yields

$$\sum_n a_n e_n \cdot \hat{\mathbf{k}} + \sum_n b_n \mathbf{f}_n \cdot \hat{\mathbf{k}} = -\sum_{\mu} A_{\mu} k_{\mu} e_{z\mu} \quad . \quad (14)$$

Applying Galerkin's procedure, we obtain an algebraic system of equations for the unknown expansion coefficients of the waveguide modes:

$$\sum_{\mu} \left( \frac{\delta_{\mu\nu}}{k_{\nu}} + \sum_n \frac{K_{n,\nu}^* K_{n,\mu}}{k_{0n}} \right) A_{\mu} = -\sum_n b_n \int_{S_c} e_{z\nu} \mathbf{f}_n \cdot \hat{\mathbf{k}} dS \quad , \quad \nu = 1, \dots, \infty \quad , \quad (15)$$

where  $\delta_{\mu\nu}$  denotes Kronecker-delta. Note that the  $b_n$  in the above equation are already determined by (6). In order to demonstrate the validity of the proposed method let us consider the structure shown in Fig. 2. This illustration presents a waveguide which extends uniformly in axial direction from  $z = 0$  to infinity. This waveguide is assumed to be short-circuited and matched at  $z = 0$  and  $z \rightarrow \infty$ , respectively. The field is excited by a point charge  $Q_0$  which is located at  $\mathbf{r}_{t0} + \hat{\mathbf{k}}z_0$ . Let us now consider the structure as an artificial cavity with its coupling aperture located at  $z = L$ . The normalized eigenfunctions  $e_{mp}$  and  $\varphi_{rs}$  are

$$e_{mp} = \sqrt{\frac{2}{k_{0mp}L(1 + \delta_{p0})}} \left[ \frac{p\pi}{k_{0mp}L} \nabla_t e_{zm} \sin\left(\frac{p\pi}{L}z\right) - \hat{\mathbf{k}} \frac{k_m^2}{k_{0mp}} e_{zm} \cos\left(\frac{p\pi}{L}z\right) \right] \quad , \quad (16)$$

$$\varphi_{rs} = \sqrt{\frac{2}{k_{0rs}L} \frac{k_r}{k_{0rs}}} e_{zr} \sin\left(\frac{s\pi}{L}z\right) \quad , \quad (17)$$

where  $m$  and  $r$  denote the transverse order of  $e_{mp}$  and  $\varphi_{rs}$ , respectively.  $p$  and  $s$  are the corresponding longitudinal orders. Inserting (16) and (17) into (15), we arrive at the

following expression for the waveguide amplitude  $A_\nu$ :

$$A_\nu = Z_0 c_0 Q_0 \frac{\sum_{s=1}^{\infty} \frac{2s\pi (-1)^s}{(\bar{k}_{0\nu s} L)^2} e_{z\nu}(\mathbf{r}_{t0}) \sin\left(\frac{s\pi}{L} z_0\right)}{\frac{1}{k_\nu} + \sum_{p=0}^{\infty} \frac{2}{k_{0\nu p}^2 L (1 + \delta_{p0})}}, \quad (18)$$

for  $\nu = 1, \dots, \infty$ . Substituting the separation conditions

$$k_{0mp}^2 = k_m^2 + \left(\frac{p\pi}{L}\right)^2, \quad (19)$$

$$\bar{k}_{0rs}^2 = k_r^2 + \left(\frac{s\pi}{L}\right)^2 \quad (20)$$

into (18) and making use of the closed-form expressions [6]

$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2 - a^2} = \frac{1}{2a^2} - \frac{\pi}{2a} \frac{\cos[(x - \pi) a]}{\sin(\pi a)} \quad (21)$$

$$\sum_{n=1}^{\infty} \frac{n \sin(nx)}{n^2 - a^2} = \frac{\pi}{2} \frac{\sin[(x - \pi) a]}{\sin(\pi a)} \quad (22)$$

yields

$$A_\nu = -Z_0 c_0 Q_0 k_\nu e_{z\nu}(\mathbf{r}_{t0}) \exp(-k_\nu L) \sinh(k_\nu z_0) \quad (23)$$

E.g., inserting the amplitudes  $A_\nu$  into (9), we get for the axial component of the electric field for  $z > L$

$$E_z = \sum_{\mu} k_\mu^2 Z_0 c_0 Q_0 e_{z\mu}(\mathbf{r}_{t0}) \sinh(k_\mu z_0) \exp(-k_\mu z) e_{z\mu}. \quad (24)$$

Note that in the cavity expansion approach the length  $L$  of the artificial cavity does not enter the expression for the waveguide field. Therefore (24) holds as long as the cavity contains the point charge which means  $z > z_0$ . The result of the modal analysis is now compared with an integration of Poisson's equation by matching the electric potential and the displacement flux at the cross section of the waveguide which contains the point charge. For  $z \neq z_0$  the electric potential  $V$  fulfills Laplace's equation  $\nabla^2 V = 0$ . Suitable series expansions for  $z < z_0$  ( $V^{(-)}$ ) and  $z > z_0$  ( $V^{(+)}$ ) are thus

$$V = \begin{cases} V^{(-)} = \sum_{n=1}^{\infty} V_n^{(-)} e_{zn} \sinh(k_n z), & z < z_0 \\ V^{(+)} = \sum_{n=1}^{\infty} V_n^{(+)} e_{zn} \exp(-k_n z), & z > z_0 \end{cases} \quad (25)$$

These expansions already satisfy the boundary condition at the shielding of the waveguide as well as at  $z = 0$  and  $z \rightarrow \infty$ . From the matching conditions on  $S$

$$V^{(-)} \Big|_{z=z_0} = V^{(+)} \Big|_{z=z_0}, \quad (26)$$

$$\hat{\mathbf{k}} \cdot (\mathbf{D}^{(+)} - \mathbf{D}^{(-)}) \Big|_{z=z_0} = Q_0 \delta(\mathbf{r}_t - \mathbf{r}_{t0}), \quad (27)$$

where  $\mathbf{D}$  denotes the displacement flux, it follows

$$V_n^{(+)} = k_n \frac{Q_0}{\epsilon_0} e_{zn}(\mathbf{r}_{t0}) \sinh(k_n z_0) \quad (28)$$

Inserting (28) into (25) and calculating  $E_z$  for  $z > z_0$  from  $E_z = -\frac{\partial}{\partial z} V^{(+)}$ , we arrive at the same result as has been previously derived with the modal expansion method as it should be. In this contribution it has been demonstrated that the computation of the dc spectral component of the electric field due to space charges can also be included in the modal analysis of aperture-coupled cavities if instead of the aperture tangential magnetic field the corresponding axial electric field is matched. It has been shown that even for  $\omega = 0$  the solenoidal electric eigenfunctions of the cavity have to be taken into account because of the waveguide coupling. Furthermore, the validity of the method has been checked by applying it to a field problem for which a solution is already well-known from another analysis.

### 3 CONCLUSIONS

In this contribution it has been demonstrated that the computation of the dc spectral component of the electric field due to space charges can also be included in the modal analysis of aperture-coupled cavities if instead of the aperture tangential magnetic field the corresponding axial electric field is matched. It has been shown that even for  $\omega = 0$  the solenoidal electric eigenfunctions of the cavity have to be taken into account because of the waveguide coupling. Furthermore, the validity of the method has been checked by applying it to a field problem for which a solution is already well-known from another analysis.

### 4 REFERENCES

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