OPTICS MEASUREMENT RESOLUTION AND BPM ERRORS

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Abstract

We study the impact of BPM resolution on optics measurements at various levels of complexity: (1) Formula linking a given distribution of BPM resolutions to the degree of precision to which any beam trajectory can be determined based on these BPM's. (2). Formula for the precision achievable in a generalized experimental scheme measuring transfer matrices in the presence of (potentially coupled) orbit errors. (3) Formula constructed from results of (1) and (2) to relate the precision of the transfer matrix measurement to the signal-to-noise ratio of the BPM system. (4) Criterion defined to summarize how well the overall optical behavior of a large modular beam transport system can be quantified. The results from (1), (2) and (3) are used to arrive at the final analytical expression providing a generic criterion on BPM resolution for such systems. Realistic examples are discussed.

1 INTRODUCTION

The purpose of this report is to present a highly accurate formulation of the criterion for BPM resolution under various optics measurement schemes.[1]

Besides monitoring the beam orbit, BPM's are collectively used in trajectory determination for feedback systems or correction programs in the control system. The trajectories in turn can be collectively used to determine the transfer matrices across a section of the beam line. This is illustrated in Figure 1. The symbol $\mathbf{m}^{\mathbf{ab}}_{\mathbf{ij}}$ stands for the ij-th transfer matrix element from point **a** to **b**, while $x^{\mathbf{p}}$ stands for the orbit vector (x,x') at the point **p**.

2 PULSE-TO-PULSE TRAJECTORY MEASUREMENT

Using the notation of Fig. 1, we study the achievable precision in determining the pulse-to-pulse trajectory at point **p** using the BPM's in beam line section **A** upstream of the unknown section. The difference between two orbits can be determined by fitting the difference in the BPM data to the known optical model of **A**:

$$\mathbf{x^B} = \begin{pmatrix} x_1^1 \\ x_1^2 \\ \vdots \\ x_1^{N_B} \end{pmatrix} = \begin{pmatrix} m_{11}^{p1} & m_{12}^{p1} \\ m_{11}^{p2} & m_{12}^{p2} \\ \vdots & \vdots \\ m_{11}^{pN_B} & m_{12}^{pN_B} \end{pmatrix} \bullet \begin{pmatrix} x_1^p \\ x_2^p \end{pmatrix} = m \bullet \mathbf{x^p},$$

$$\mathbf{x^p} = m^{-1} \bullet \mathbf{x^B} \qquad (1.1)$$

where N_B is the number of BPM's used. The matrix inverse represents the least square fit. The covariance error matrix for the fitted orbit vector at **p**, $<\delta_x{}^{\mathbf{p}i}\delta_x{}^{\mathbf{p}j}>$, can be derived, using symplectic conditions, as a function of the optics and the resolution $\sigma_B{}^q$ for the BPM's, with q indexing the BPM:

$$\sigma_X^{p1\,2} = \left\langle \delta_X^{p1} \cdot \delta_X^{p1} \right\rangle = \frac{2}{N_B} \cdot \frac{\left\langle m_{12}^{pa} \right\rangle_s^2}{\left\langle m_{12}^{aa} \right\rangle_s^2},$$

$$\sigma_X^{p2\,2} = \left\langle \delta_X^{p2} \cdot \delta_X^{p2} \right\rangle = \frac{2}{N_B} \cdot \frac{\left\langle m_{11}^{pa} \right\rangle_s^2}{\left\langle m_{12}^{aa} \right\rangle_s^2},$$

$$S^{p12} = \left\langle \delta_X^{p1} \cdot \delta_X^{p2} \right\rangle = \frac{-2}{N_B} \cdot \frac{\left\langle m_{11}^{pa} \cdot m_{12}^{pa} \right\rangle_s}{\left\langle m_{12}^{aa} \right\rangle_s^2},$$

$$\left\langle m_{12}^{aa} \right\rangle_s^2 = \sum_{a^i=1}^{N_B} \sum_{a^j=1}^{N_B} \left(\frac{m_{12}^{a^i a^j}}{\sigma_B^{a^i} \sigma_B^{a^j}} \right)^2,$$

$$\left\langle m_{12}^{pa} \right\rangle_s^2 = \sum_{a^j=1}^{N_B} \left(\frac{m_{12}^{pa^j}}{\sigma_B^{a^j}} \right)^2,$$

$$\left\langle m_{11}^{pa} \cdot m_{12}^{pa} \right\rangle_s = \sum_{a^j=1}^{N_B} \left(\frac{m_{11}^{pa^j} m_{12}^{pa^j}}{\sigma_B^{a^j} \sigma_B^{a^j}} \right),$$

$$\left\langle m_{11}^{pa} \right\rangle_s^2 = \sum_{a^j=1}^{N_B} \left(\frac{m_{11}^{pa^j}}{\sigma_B^{a^j}} \right)^2. \qquad (1.2)$$

This result can be used in feedback systems or other control program designs.

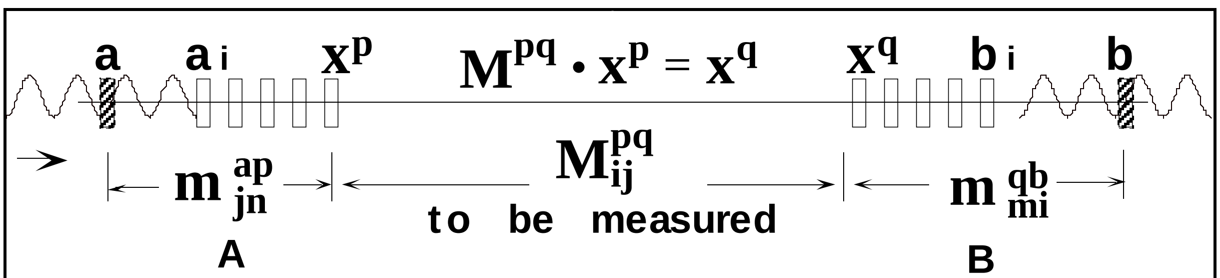

FIGURE 1

Partitioning the Double Sum

In many cases discussed below we can partition the BPM's into subgroups and simplify the double sum in Eqn. 1.3. These subgroups can be identical cells or all the BPM's identically located in each cell. The double sum then is reduced to

$$\frac{1}{G^2}\cdot\left\langle m_{12}^{aa}\right\rangle^2 = \sum_{k=1}^{G}\left\langle m_{12}^{aa}\right\rangle_k^2 + \sum_{\substack{m>n\\ m=1,n=1}}^{G}\sum_{a=1}^{N_m}\sum_{b=1}^{N_n}2\left(m_{12}^{ab}\right)^2. \tag{1.4}$$

G above is the total number of subgroups, indexed by m and n. The subscript k indicates double sum only within a subgroup.

Simple Rule of Thumb

The sums in Eqn. 1.3 are actually very easy to calculate.[1] If one wants even more immediate estimates, the following rules of thumb can be a substitute. Notice that the last three equalities break down for small number of BPM's.

$$\begin{aligned}
\left\langle m_{12}^{aa}\right\rangle^2 &= \left\langle\beta\beta\right\rangle_{\sin}^2 = \frac{1}{2N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\beta^i\cdot\beta^j\cdot\sin^2\left(\varphi^i-\varphi^j\right),\\
\left\langle m_{11}^{pa}\right\rangle^2 &\xrightarrow{N>>1}\frac{\gamma_p\langle\beta\rangle}{2} = \frac{\gamma_p}{2N}\sum_{j=1}^{N}\beta^j,\\
\left\langle m_{12}^{pa}\right\rangle^2 &\xrightarrow{N>>1}\frac{\beta_p\langle\beta\rangle}{2} = \frac{\beta_p}{2N}\sum_{j=1}^{N}\beta^j,\\
\left\langle m_{11}^{pa}\cdot m_{12}^{pa}\right\rangle &\xrightarrow{N>>1}\frac{-\alpha_p\langle\beta\rangle}{2} = \frac{-\alpha_p}{2N}\sum_{j=1}^{N}\beta^j .
\end{aligned} \tag{1.5}$$

The subscript p labels the observation point **p**.

3 TRANSFER MATRIX MEASUREMENT

A scheme for measuring the unknown transfer matrix M^{pq}_{ij} is devised in Fig. 1. A total of N_O trajectories are sent through beam line sections **A** and **B**, where the orbit vectors are determined in the fashion discussed above at observation points **p** and **q**. These two sets of orbit vectors are sufficient for unfolding the unknown M^{pq}_{ij}. This scheme is better than the commonly adopted method relying only on knowledge of upstream kickers, in that it is immune to kicker errors and incoming orbit/energy jitters, that the beam line structure affords more exact error analysis, and that the flexibility in expanding the upstream section frees us from the limit on overall precision occurring otherwise[1]. The fitting problem now takes on the form

$$\left(M_{11}^{pq}\quad M_{12}^{pq}\right)\bullet\begin{pmatrix} x_1^{p1} & x_1^{p2} & \dots & \dots & x_1^{pNo}\\ x_2^{p1} & x_2^{p2} & \dots & \dots & x_2^{pNo}\end{pmatrix} = \left(x_1^{q1}\quad x_1^{q2}\quad \dots\quad \dots\quad x_1^{qNo}\right),$$

$$\left(M_{21}^{pq}\quad M_{22}^{pq}\right)\bullet\begin{pmatrix} x_1^{p1} & x_1^{p2} & \dots & \dots & x_1^{pNo}\\ x_2^{p1} & x_2^{p2} & \dots & \dots & x_2^{pNo}\end{pmatrix} = \left(x_2^{q1}\quad x_2^{q2}\quad \dots\quad \dots\quad x_2^{qNo}\right). \tag{2.1}$$

Fitting for M^{pq}_{ij} is more involved now that the orbit vectors on both sides of Eqn. 2.1 have random errors, most likely coupled in the manner of Eqn. 1.2. Similar problem involving uncorrelated orbit errors in normalized coordinates has been addressed[2]. They correspond to eigenvectors of the covariance matrix constructed as follows:

$$\mathbf{C}_{ij} = \sum_{k=1}^{N_o} z_i^k\cdot z_j^k,\quad \mathbf{C}_{ij}\cdot\mathbf{N}_j = \lambda\,\mathbf{N}_i\ ,$$

$$\mathbf{u_1} = \frac{\mathbf{N}}{|\mathbf{N}|}\ ,\quad |\mathbf{u_r}| = 1,\ \ r = 2,\dots .n. \tag{2.2}$$

The $z^k{}_i$'s are the orbit vectors normalized by the uncoupled errors. The eigenvectors $\mathbf{N}_i$ contain the fitted transfer matrix elements, which are then normalized to $\mathbf{u}_i$'s. The error covariance for the $\mathbf{u}_i$'s is given by

$$\left\langle(\delta\mathbf{u_1})_i(\delta\mathbf{u_1})_j\right\rangle = \sum_{r=2}^{n}\frac{\lambda_r+\lambda}{(\lambda_r-\lambda)^2}(\mathbf{u_r})_i(\mathbf{u_r})_j\ . \tag{2.3}$$

To make Eqn. 2.3 applicable, we need to take the following steps.

- Diagonalization: We find the transformations diagonalizing the orbit error covariance matrices in both upstream and downstream sections. This is accomplished with symplectic matrices of the form

$$O_{p,q} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & m\sqrt{\dfrac{A_{p,q}}{D_{p,q}}}\\ \pm\sqrt{\dfrac{D_{p,q}}{A_{p,q}}} & 1\end{pmatrix}, \tag{2.4}$$

$$A_{p,q} = \left\langle\delta x_1^{p,q}.\delta x_1^{p,q}\right\rangle\ ,\quad B_{p,q} = \left\langle\delta x_1^{p,q}.\delta x_2^{p,q}\right\rangle\ ,\quad D_{p,q} = \left\langle\delta x_2^{p,q}.\delta x_2^{p,q}\right\rangle\ .$$

In doing this we introduce extra couplings among the orbit vectors at **p** and **q**.

- Application of Eqn. 2.3**:** This gives the error covariance between the fitted matrix elements in the diagonalized coordinates.
- Cross coupling between rows: The two equations of Eqn. 2.1 appear uncorrelated. They nonetheless are coupled through sharing the same set of incoming orbits. This coupling has nontrivial effects when we restore to the undiagonalized coordinates. This effect, not addressed by Eqn. 2.3, has to be calculated.
- Un-normalizing the unit vectors $\mathbf{u}_i$: This gives the covariance in $\mathbf{N}_i$'s.
- Un-diagonalization: This gives the final error covariance in the physical coordinates, summarized as follows:

$$\left\langle \delta M_{ij}^{pq} \cdot \delta M_{kl}^{pq} \right\rangle = \sum_m \sum_n \sum_r \sum_s T^{mnrs}$$

$$T^{mnrs}$$

$$= \left(O_q\right)^{-1}_{im} \cdot \left(O_q\right)^{-1}_{kr} \cdot \left\langle \delta M'^{pq}_{mn} \cdot \delta M'^{pq}_{rs} \right\rangle \cdot \left(O_p\right)_{nj} \cdot \left(O_p\right)_{sl} . \tag{2.5}$$

One quantity $\delta^{qi}{}_{Em}$, defined by

$$\delta^{q1}_{Em} = M^{pq}_{11} \cdot \delta^{p1}_{Xm} + M^{pq}_{12} \cdot \delta^{p2}_{Xm} - \delta^{q1}_{Xm} \ ,$$
$$\delta^{q2}_{Em} = M^{pq}_{21} \cdot \delta^{p1}_{Xm} + M^{pq}_{22} \cdot \delta^{p2}_{Xm} - \delta^{q2}_{Xm} \ ,$$
$$m = 1, 2, \ldots N_O , \tag{2.6}$$

stands out in the final expression. Notice that $\delta^{qi}{}_{Em}$ has an index m for the trajectory number. It represents the error at the exit point **q** when the difference between the exit orbit and the properly propagated entrance orbit from **p** is calculated. The error covariance in the fitted matrix elements then takes on an intuitive form:

$$\left\langle \delta M^{pq}_{ij} . \delta M^{pq}_{km} \right\rangle = \frac{1}{N_O} \cdot \frac{\left\langle \delta^{qi}_E \cdot \delta^{qk}_E \right\rangle}{\left\langle x^p_j \cdot x^p_m \right\rangle_{(d)}} , \quad i, j, k, m = 1, 2 .$$

$$\left\langle x^p_j \, . \, x^p_m \right\rangle_{(d)} = \begin{cases} \left\langle x^p_j \, . \, x^p_m \right\rangle \cdot \left(1 - R^2_p\right), & j = m \\ \left\langle x^p_j \, . \, x^p_m \right\rangle \cdot \left(1 - R^{-2}_p\right), & j \neq m \end{cases} ,$$

$$R^2_p = \frac{\left\langle x^p_1 \, . \, x^p_2 \right\rangle^2}{\left\langle x^p_1 \, . \, x^p_1 \right\rangle \cdot \left\langle x^p_2 \, . \, x^p_2 \right\rangle} . \tag{2.7}$$

4 OVERALL FORM FACTOR

Combining Eqns. 1.3 and 2.7, we can calculate the overall error covariance in the fitted matrix elements in terms of the signal-to-noise ratio of the BPM's. We need to use the generalized symplectic condition:

$$m^{p\,ai}_{12} = -\frac{P_p}{P_{ai}} \left(m^{p\,ai}\right)^{-1}_{12} = -\frac{P_p}{P_{ai}} m^{ai\,p}_{12} ,$$
$$m^{p\,ai}_{11} = -\frac{P_p}{P_{ai}} \left(m^{p\,ai}\right)^{-1}_{22} = -\frac{P_p}{P_{ai}} m^{ai\,p}_{22} , \tag{3.1}$$

in case the momenta are different at **p** and **q**. This allows us to propagate all the orbits from **p** to **q.** The overall error covariance is then given by

$$\left\langle \delta M^{pq}_{ij} \cdot \delta M^{pq}_{km} \right\rangle =$$

$$2 \cdot \frac{1}{N_O} \cdot S^{jm}_B \cdot \frac{1}{T^{jm}_{(d)}} \cdot \left[\frac{1}{N_{B_q}} \cdot \mathcal{M}^i_b \cdot \mathcal{M}^k_b + \right.$$

$$\left. \frac{1}{N_{B_p}} \cdot \mathcal{M}^i_a \cdot \mathcal{M}^k_a \cdot \left(\frac{P_p}{P_q}\right) \right], \quad i, j, k, m = 1, 2 .$$

$$R^2_p = \frac{\left\langle x^p_1 \, . \, x^p_2 \right\rangle^2}{\left\langle x^p_1 \, . \, x^p_1 \right\rangle \cdot \left\langle x^p_2 \, . \, x^p_2 \right\rangle} ,$$

$$S^{jm}_B = \frac{\sigma^2_B}{\sigma^{pj}_O \cdot \sigma^{pm}_O} , \quad T^{jm}_{(d)} = \begin{cases} \left(1 - R^2_p\right), & j = m \\ \left(1 - R^{-2}_p\right), & j \neq m \end{cases} ,$$

$$\mathcal{M}^1_a = \frac{\left\langle m^{qa}_{12} \right\rangle}{\left\langle m^{aa}_{12} \right\rangle} , \quad \mathcal{M}^2_a = \frac{\left\langle m^{qa}_{11} \right\rangle}{\left\langle m^{aa}_{12} \right\rangle} ,$$

$$\mathcal{M}^1_b = \frac{\left\langle m^{qb}_{12} \right\rangle}{\left\langle m^{bb}_{12} \right\rangle} , \quad \mathcal{M}^2_b = \frac{-\left\langle m^{qb}_{11} \right\rangle}{\left\langle m^{bb}_{12} \right\rangle} . \tag{3.2}$$

The physical significance of these quantities deserves some elaboration:

1. Factors determined by experimental parameters:
- $S_B{}^{jm}$ is the generalized signal to noise ratio. It may take on a dimension of meter or meter^2 in some cases.
- $T_{(d)}{}^{jm}$ characterizes the position-angle coupling at the observation point **p**, nearly inevitable in real experiments. When R_p=0, this term makes some of the correlation terms disappear.
- N_O is the sample size, i.e., the number of orbits used.

2. Factors determined by machine parameters:
- $N_{p,q}$ are the number of BPM's used to determine each trajectory at the observation points **p** and **q** respectively. It's evident by Eqn. 3.1 that the overall precision can not be improved indefinitely by increasing the number of BPM's only on one side of the measured section.
- $P_{p,q}$ are the momentum values at the observation points **p** and **q**.
- $\mathcal{M}_{a,b}{}^{1,2}$ are the RMS ratios defined in Eqn. 1.3. Their evaluation is easier than appears.(1) Notice the minus sign in the last equation.

For a BPM system designer, these quantities translate into other machine specifications and have to be taken into account in optimizing the performance. For example, N_O is limited by the speed of the BPM electronics and operation/control interface, $S_B{}^{jm}$ is limited by the beam pipe radius and transfer properties all around the machine, $\mathcal{M}_{a,b}{}^{1,2}$ are bound by optical or experimental conditions, while everything else has to conform to cost restrictions. But Eqn. 3.2 does take the guesswork out of the design so far as optical requirements are concerned. All analytic formulas presented above have been numerically verified.

ACKNOWLEDGMENTS

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REFERENCES

[1] Complete detail, more specific formulas and numerical examples can be found in ‘Optics-Driven Design Criteria for BPM's’, Y. Chao CEBAF TN-93-073 (1993).
[2] Lohse, T. and Emma, P., "Linear Fitting of Beam Orbits and Lattice Parameters," SLAC-CN-371 (1989).