TWO EXAMPLES OF INTEGRABLE SYSTEMS WITH ROUND COLLIDING BEAMS

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Abstract

The work is devoted to analytical study of application of integrable systems to round colliding beams, aiming at enhancement of the beam-beam limit. Two examples of "integrable" beam-beam forces are presented, relevant to round counter beams with special density distributions. In such systems all the resonances will vanish, hence the beam-beam effects can be suppressed, and the intensities of colliding beams may be strongly increased, at least in the "weak-strong" case.

1 INTRODUCTION

The concept of "Round Colliding Beams" (RCB) is considered as a possibility to reach higher luminosity and to improve beam stability in colliders ([1] and references therein). The essential conditions of the RCB are: equal horizontal and vertical emittances $\varepsilon_x = \varepsilon_y = \varepsilon$; equal horizontal and vertical beta-functions at the Interaction Point (IP) $\beta_x^* = \beta_y^* = \beta^*$; equal horizontal and vertical tunes $\nu_x = \nu_y = \nu$. The rotational symmetry of the kick from the round opposite beam, complemented with the $X - Y$ symmetry of the betatron transfer matrix between the collisions, result in an additional integral of motion $\mathcal{M} = xy' - yx'$, *i.e.* the longitudinal component of particle's angular momentum.

Thus, the transverse motion becomes equivalent to a one-dimensional (1D) motion [2]. Resulting elimination of all betatron coupling resonances is of crucial importance, since they are believed to cause the beam lifetime degradation and blow-up. Reduction to 1D motion makes impossible the diffusion through invariant circles. Although this 1D motion has more "regularity" in comparison with a general 2D motion, with the time-dependent Hamiltonian it is still stochastic in general. What we need here to make the motion regular, is *to construct one more integral of motion, valid for any value of the angular momentum* $\mathcal{M}$. At first glance, it is not evident, that we can find the needed forces (among those physically feasible), especially when we deal with the fields of the counter beam. But solutions exist [3], and we present two interesting examples, which may be already useful for practice.

2 EXAMPLE 1: INTEGRABLE BEAM-BEAM KICK

Let us take a drift space with the unity length (for simplicity) followed by an axially symmetric thin lens, as a representation of the angular-momentum-preserving linear optics in between the IPs, and the radial beam-beam kick. The 2D map for particle trajectory displacements x, y and slopes x', y' through such a period is:

$$\begin{aligned} \overline{x} &= x + x' \\ \overline{y} &= y + y' \\ \overline{x}' &= x' + \overline{k}_x \\ \overline{y}' &= y' + \overline{k}_y, \end{aligned} \tag{1}$$

where $k_x = \dfrac{x}{r}k(r)$, $k_y = \dfrac{y}{r}k(r)$, and $r = \sqrt{x^2+y^2}$. Due to conservation of the angular momentum, the motion is reducible to 1D. Previously we reported on existence of invariants of this map in the particular case of 1D motion ([4] and referencs therein). This corresponds in (1) to $x = x' = 0$, or $y = y' = 0$, or generally, to any meridianal trajectory with $\mathcal{M} = xy' - yx' = 0$. With x, x' lying in the plane of such a trajectory, the desired integrals of motion may be sought among these invariants:

$$\begin{aligned} &\mathcal{I}(x,x') = (a_2x^2 + a_1x + a_0)(x'+x)^2 \\ &+(a_1x^2 + b_1x + b_0)(x'+x) + a_0x^2 + b_0x\,, \end{aligned} \tag{2}$$

and the kick function k must have the form:

$$k(x) = -2x - \frac{a_1x^2 + b_1x + b_0}{a_2x^2 + a_1x + a_0}\,. \tag{3}$$

Here the 5 coefficients are arbitrary parameters of the kick force.

Turning back to the general case $\mathcal{M} = M \neq 0$, we can use the generic form of 1D invariant (2) for construction of an axially symmetric invariant involving only r, r' as dynamic variables. The kick function (3) is now understood as a radial kick $k(r)$, and we observe that only the case $b_0 = a_1 = 0$ is practically interesting, otherwise $k(r)$ would have singularities at $r = 0$. The Courant–Snyder terms with a_0, b_1 in (2) give a clue to the form of the axially-symmetric invariant, to be tried for any value M of the angular momentum $\mathcal{M} = xy' - yx'$ (certainly valid at $M = 0$):

$$\begin{aligned} \mathcal{I}_M(r,r') = (a_2r^2 + a_1r + a_0)\Big((r'+r)^2 + \frac{M^2}{r^2}\Big) \\ +(a_1r^2 + b_1r + b_0)(r'+r) + a_0r^2 + b_0r. \end{aligned} \tag{4}$$

The variables here are changed to r, r', use has been made of the following relations: $r' = (xx' + yy')/r$, $x'^2 + y'^2 = \big((rr')^2 + (xy'-yx')^2\big)/r^2 = r'^2 + M^2/r^2$.

Rewriting accordingly the map (1) in terms of (r, r'):

$$\begin{aligned} \overline{r} &= \sqrt{r^2 + 2rr' + r'^2 + M^2/r^2}, \\ \overline{r}' &= \Big(r'(r'+r) + \frac{M^2}{r^2}\Big)\frac{1}{\overline{r}} + k(\overline{r}), \end{aligned} \tag{5}$$

we apply this transformation to (4). The invariance relation $\mathcal{I}_M(r,r') = \mathcal{I}_M(\overline{r},\overline{r}')$ then yields: $a_1 = b_0 = 0$. Thus we find the desired integral of motion which holds at any constant value M of $\mathcal{M}$:

$$\mathcal{I}_M(r,r') = (a_2 r^2 + a_0)(r'+r)^2 + b_1 r(r'+r) + a_0\Big(r^2 + \frac{M^2}{r^2}\Big). \quad (6)$$

The corresponding radial kick function

$$k(r) = -2r - \frac{b_1 r}{a_0 + a_2 r^2}\,. \quad (7)$$

has only 3 free parameters, just in accord with our assumption that the integrable systems for RCB form a subset of all 1D integrable systems.

In the present context we interpret the 2nd term in (7) as the beam-beam kick, while the 1st term together with the drift length form the linear optics in between the IPs. The optics appears to be a 90° lattice in both X and Y planes, with the matrix of the period:

$$T_x = T_y = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (8)$$

In order to return to the physical units in the result one should replace 1 to β^*, and -1 to $-1/\beta^*$.

Relevance of the solution to the beam-beam kick force from a short (relativistic!) opposite bunch (practically, with the length $\ll \beta^*$), can be seen from the conformity of both $r \to 0$ and $r \to \infty$ limits of the fraction in (7) to the realistic beam-beam kick behaviour. We may put $a_0 = 1$, $a_2 > 0$, relate $a_2^{-1/2}$ to the radial beam size Δ, and express b_1 via the beam-beam parameter ξ: $b_1 = 4\pi\xi$. Specifically, the transverse density distribution in the counter bunch

$$\mathrm{d}n \propto N \frac{2\pi r \mathrm{d}r}{(1 + (r/\Delta)^2)^2} \quad (9)$$

exactly corresponds to the kick (7).

In practice, implementation of this solution in a RCB scheme requires short colliding bunches with radial distributions close to (9), a linear optics with equal transfer matrices in x and y planes, and with equal betatron phase advances of 90° in between the IPs. Integrability of the resulting dynamics will show in regularity of motion which will be bounded by closed invariant curves $\mathcal{I}_M(r,r') = const$, and free from resonance islands throughout the linear stability range, *i.e.* for the beam-beam parameter $|\xi| < 1/2\pi$.

3 EXAMPLE 2: A SPECIAL LONGITUDINAL DISTRIBUTION

In the previous example we dealt with a short nonlinear kick, the time dependence was represented by the delta-function. Now we turn to a continuous-time dependence of the nonlinear force, and present a dynamical system with two invariants, which can be derived by means of usual accelerator theory tools. Let us take the 1D equation of particle's motion in an accelerator:

$$x'' + g(s)x = F(x,s), \quad (10)$$

where $g(s)$ is the focusing function and $F(x,s)$ is an arbitrary force. This equation can be simplified by using the betatron phase $\psi = \int ds/\beta(s)$ instead of s and changing the physical variable x to the normalized variable $X = x/\sqrt{\beta(s)}$:

$$X'' + X = \beta^{3/2} F(X\sqrt{\beta}, s(\psi)). \quad (11)$$

The force due to round counter beam with the transverse Gaussian distribution can be presented now in the factorized form, thus separating its dependence on the transverse and longitudinal coordinates:

$$F_{rb} = -\frac{2Ne^2}{\gamma mc^2}\,\frac{1 - \exp(-r^2/2\beta\varepsilon)}{r/\sqrt{\beta}}\,\frac{f(\delta - 2s)}{\sqrt{\beta}}\,, \quad (12)$$

Here ε is the emittance of the opposite beam, f is the longitudinal distribution of counter beam ($\int f d\delta = 1$), δ is the longitudinal position of the test particle in the weak bunch with respect to the bunch center. The "time" $s = 0$ corresponds to the moment when the central test particle ($\delta = 0$) meets the center of the strong bunch.

The equation of particle motion in the interaction region in terms of $r = \sqrt{x^2 + y^2}$ is:

$$r'' + g(s)r = F_{rb} + M^2/r^3, \quad (13)$$

where the last term means the "centrifugal" force.

Now let us consider a case when the weak bunch of the test particles is short with respect to the beta function at the IP and we can put $\delta = 0$, and at the same moment, the longitudinal charge distribution of the strong bunch is proportional to the inverse β-function: $f(2s) = C/\beta(s)$. For an interaction region which is free of focusing we have: $\beta(s) = \beta^* + s^2/\beta^*$, and the perfect distribution is:

$$f(s) = \frac{C}{1 + (s/2\beta^*)^2}, \quad (14)$$

where C is a constant, β^* is the β-function value at the IP.

After substitution of the normalized variable $R = r/\sqrt{\beta(s)}$ and replacement of s by the phase ψ, one gets:

$$R'' + R = \frac{M^2}{R^3} - C\frac{2Ne^2}{\gamma mc^2}\frac{1 - \exp(-R^2/2\varepsilon)}{R}. \quad (15)$$

One can see, that the force in this equation does not depend on time, and therefore, this 1D equation is integrable. The coordinate dependence on time can be found using conventional 1D formulas.

The trick with obtaining the time-independent force is related with the fact, that the 'centrifugal' force is invariant under substitution of new variables and changing 'time' to the betatron phase. It is easy to see, that the force in (15) has one zero for a counter beam with opposite sign of electric charge.

If one wants to realize this strategy in practice, then one has to create the longitudinal distribution in the form of Eq.(14) over a wide range of s, and to make it zero outside. Next step is to make a linear transformation with the phase advance of $2\pi m$, m is an integer[1], to the next interaction region. Having reversed the sign of β-function derivative, we enter again in the interaction region and see the same counter beam with the same distribution (and so on periodically), we have a system with the needed equation of motion.

Of course, it is difficult to create a longitudinal distribution in accurate approximation to the perfect bunch shape over entire length of the strong bunch, including its tails. But apparently, small deviations of the distribution from the inverse β-function for $s \gg \beta^*$, will cause negligible perturbation. Approximation of the perfect longitudinal distribution (14) at small s by a Gaussian $\exp(-s^2/2\sigma^2) \approx 1 - s^2/2\sigma^2$, leads to a recipe for an optimum length of a longitudinally-Gaussian strong bunch: $\sigma = \sqrt{2}\beta^*$.

One can see, that the Gaussian shape of the betatron distribution is not important here; the distribution may be any smooth function of r, s, provided that its dependence on R and s can be factorized.

Another essential feature of this solution is that the working point of this system is near the half-integer resonance, when the number of interaction points is odd, and near the integer resonance, when this number is even. So, perturbations of the arcs of collider determine the permissible distance of the working point from the resonance, and consequently, determine the accuracy of the conservation of integrals (this situation is common for integrable systems: perturbations almost always lead to small stochasticity in nearly integrable systems, the point is in allowable values of perturbations).

One more remark is needed. The weak bunch has a small longitudinal size in our dynamical system (while the longitudinal distribution of the strong bunch is taken proportional to the inverse β-function). If its length is not small in comparison with the β-function, then the force becomes time-dependent for particles with large energy off-sets and deviations δ even in the normalized variables. Importance of this modulation was checked by simulation [5].

A formal construction of the integrable distribution for the above example can be found in [3].

4 CONCLUSION

The paper presents new ways to improve single particle stability in colliders. The essence of these ways is obtaining integrability of the particles' dynamics with proving additional integrals of the particle motion. For example, if the "round colliding beams" conditions are fulfilled then the longitudinal component of the angular momentum is the invariant for colliding beams. In some particular cases of the RCBs, additionally to the angular momentum $\mathcal{M}$, we can find one more invariant which is quadratic in momentum, and holds for any value of $\mathcal{M}$. These two integrals of motion suffice for a non-autonomous 2D dynamical system to be integrable.

The both RCB examples above deal with what is called a "weak-strong" beam-beam model: study of motion of a test particle affected by a strong nonlinear fields of a strong counter beam. In the both recipes the arc lattice in between the IPs must be perfectly linear. The 1st example with a short counter bunch requires a special transverse distribution in the strong bunch for integrability. In the 2nd example, the strong bunch length is of the order of β^*. Here, we can choose an "inverse β-function" longitudinal bunch profile for reduction of motion to a 1D autonomous dynamics.

The proposed integrable systems with globally regular motion and without any beam-beam blow-up threshold have strengthened the concept of round colliding beams. They were tested in simulations [1, 5] against perturbations inevitably present in a real machine. The beam emittance growth becomes mostly determined by the arc lattice nonlinearities and imperfections, so we believe that it will be possible to achieve a higher luminosity by reducing the impact of nonlinear lattice resonances.

5 REFERENCES

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[1] Actually, the both x and y phase advances of π are also acceptable. We obtain the same motion for r, r' due to symmetry of potential of this motion. In this case we always stay near the integer resonance.