PATH INTEGRAL COMPUTATION OF LASLETT COEFFICIENTS IN PIPES WITH ARBITRARY TRANSVERSE GEOMETRY

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Abstract

An efficient (accurate and fast) numerical framework is introduced for computing the (incoherent) betatron normal mode Laslett coefficients for beam-liner with completely general transverse geometry.

1 INTRODUCTION

In previous papers[1][2] we introduced a general framework for computing the normal mode Laslett coefficients for beam liners in terms of the associated complex (2D) potential. In this communication we extend the previous result to the most general case where the (transverse) liner geometry is so complicated that the (2D) potential cannot be computed in analytic form. The extension is based on (a suitable generalization of) Cauchy integral formula for computing the Lasletts without using derivatives, and an application of Monte-Carlo Wiener-Ito stochastic integration for solving Poisson equation.

2 NORMAL MODE LASLETT COEFFICIENTS

The betatron normal mode incoherent[1] Laslett coefficients for beam liners possibly surrounded by a co-axial magnetic yoke can be conveniently computed from an auxiliary complex potential $\bar{\Psi}$:[1][2]

$$\phi^{(im.)} - \beta_0 A_z^{(im.)} = 2\Lambda \; Re \; \bar{\Psi}(\bar{z}, \bar{z}_b, \bar{z}_b^*), \qquad (1)$$

as:

$$\epsilon_{1,2}^{(inc.)} = \pm \frac{1}{2} \left| \ddot{\bar{\Psi}} \right|_{\bar{z}=\bar{z}_b=\bar{z}_{eq.}} . \qquad (2)$$

In (1) $\bar{z} = L^{-1}(x+iy)$ and $\bar{z}_b = L^{-1}(x_b+iy_b)$ denote the (complex, scaled) field-point and source-point transverse positions, $\bar{z}_{eq.} = L^{-1}(x_{eq.} + iy_{eq.})$ is the beam center of charge (complex, scaled) equilibrium position, L is a problem-dependent scaling length (e.g., the pipe diameter), and $\phi^{(im.)}$, $A_z^{(im.)}$ are the electric and magnetic *image potentials*, which are obtained by subctracting from the full potentials the free-space terms, and dots indicate derivation w.r.t. the argument.[2]

Unfortunately, Eq. (2) seems useless whenever explicit (analytic) solutions of Laplace equation are not available.

3 THE SCHWARTZ FORMULA

Whenever only a *numerical* solution for the (real valued) potential ϕ is possible *at a discrete set of points*, the Laslett coefficients can *still* be efficiently computed using (2) by numerical evaluation of Schwartz formulas[3][2]:

$$\ddot{\bar{\psi}}(z_0) \; = \; \frac{2}{\pi} \int_0^{2\pi} \frac{1}{(z-z_0)^2} \, \phi[Re(z), Im(z)] \, d\theta \qquad (3)$$

where $z = z_0 + Re^{i\theta}$ is any circle embracing z_0, and $f(z)$ is analytic within and on the aforementioned circle. Computing the Lasletts in terms of *integrals* instead of *derivatives* of $\bar{\Psi}$ is indeed nice, since, loosely speaking, for the same prescribed accuracy, this requires knowledge of *much fewer values* of the numerically evaluated potential. However, in order to use (3), one needs to know the potential at a suitable number of points *on a circle*, whereas *most* available numerical methods yield values of ϕ arranged on *noncircular* mesh grids. This suggests resorting to a numerical algorithm which *does not* rely to *any* prior meshing to find a numerical solution for ϕ.

4 POTENTIALS FROM WIENER INTEGRALS

The connection among Wiener processes, functional integrals and Poisson's equation is well known in probability theory.[4] Accordingly, the solution of the Dirichlet problem:

$$\begin{cases} \nabla_t^2 \Phi(x,y) = -g(x,y), & (x,y) \in \mathcal{D}, \\ \Phi(x,y) = 0, & (x,y) \in \partial\mathcal{D}. \end{cases} \qquad (4)$$

can be written[4][3]:

$$\Phi(x,y) = E \left[\int_0^\tau g\left[w_x(t), w_y(t)\right] \, dt \right], \qquad (5)$$

where $E[\cdot]$ represents an *expectation value* with respect to the probability measure associated to the Wiener processes (random walks) (w_x, w_y) starting at (x,y) at time $t = 0$, and (first) crossing the boundary $\partial\mathcal{D}$ at $t = \tau$.

Monte Carlo Implementation

The expectation value in (5) is recognized to be a Wiener Integral, which can be computed using Monte Carlo

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[1]For a strictly relativistic ($\beta_0 = 1$) beam, Eq. (2) also gives the coherent betatron normal mode Laslett coefficients in the non-penetrating magnetic-field regime.[2]

[2]Schwartz formulas can be derived from Cauchy integral theorem, by exploiting the well known fact that the analytic complex potential is uniquely determined, up to an irrelevant constant, from its real part alone.

[3]Equation (4) can be extended to the case where $g(x,y)$ is a line source by a suitable limiting process.

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methods.[5] Accordingly, we introduce a piecewise-linear, $\Delta-$time-step discretized *Euler approximation*[6] for the Wiener processes involved:[4]

$$\begin{cases} w_x(t_{k+1}) = w_x(t_k) + \sqrt{2}\,\delta_x \quad , \quad w_x(0) = x, \\ w_y(t_{k+1}) = w_y(t_k) + \sqrt{2}\,\delta_y \quad , \quad w_y(0) = y, \end{cases} \tag{6}$$

where $t_k = k\Delta$, and δ_x, δ_y are independent gaussian random variables with moments:

$$E(\delta_x) = E(\delta_y) = 0 \quad , \quad E[(\delta_x)^2] = E[(\delta_y)^2] = \Delta. \tag{7}$$

A random walk is created, using (6),(7), and evolved until, at $t = t_Q$, it has (first) gone beyond $\partial\mathcal{D}$, viz.:

$$(w_x(t_Q), w_y(t_Q)) \notin \mathcal{D},$$
$$(w_x(t_k), w_y(t_k)) \in \mathcal{D}, \;\; 0 \le k < Q. \tag{8}$$

The exit time τ is computed from:

$$\frac{\tau - t_{Q-1}}{\Delta} =$$
$$= \left\{ \frac{[x_e - w_x(t_{Q-1})]^2 + [y_e - w_y(t_{Q-1})]^2}{[w_x(t_Q) - w_x(t_{Q-1})]^2 + [w_y(t_Q) - w_y(t_{Q-1})]^2} \right\}^{-1/2} \tag{9}$$

where the coordinates of the *exit point* (x_e, y_e) are the (only) solution in $x_e \in [min\{w_x(t_{Q-1}), w_x(t_Q)\}, max\{w_x(t_{Q-1}), w_x(t_Q)\}]$ of the system between the (rectilinear) random walk between t_{Q-1} and t_Q:

$$y_e - w_y(t_{Q-1}) = \frac{w_y(t_Q) - w_y(t_{Q-1})}{w_x(t_Q) - w_x(t_{Q-1})}\,[x_e - w_x(t_{Q-1})], \tag{10}$$

and the equation describing (locally) the $\mathcal{D}-$domain boundary curve, $y_e = f(x_e)$. For each realization (labeled by a superfix j) of the random path we can compute the random variable:

$$\phi^{(j)}(x,y) = \Delta \sum_{k=0}^{Q_j - 1} g\left[w_x^{(j)}(t_k), w_y^{(j)}(t_k)\right] +$$
$$+ (\tau^{(j)} - t_{Q_j - 1})\; g\left[x_e^{(j)}, y_e^{(j)}\right], \tag{11}$$

and the associated first and second order moments $\mu_1(M,\Delta)$ and $\mu_2(M,\Delta)$ where M is the total number of paths.

The potential in (5) can be accordingly evaluated as a double limit:

$$\Phi(x,y) = \lim_{\Delta \longrightarrow 0} \lim_{M \longrightarrow \infty} \mu_1(M,\Delta). \tag{12}$$

Estimating $\Phi(x,y)$ by $\mu_1(M,\Delta)$ will produce, for any finite Δ and M, both *i)* a systematic error ϵ_{sys} due to discretization, for which:[6]

$$\epsilon_{sys}(\Delta) \sim \mathcal{O}(\Delta), \tag{13}$$

[4] In the following we refer to Wiener processes with diffusion coefficient $D = 1$, and use a dimensionless time variable.

and *ii)* a statistical error, ϵ_{stat} which in view of the central limit theorem is asymptotically gaussian, with zero average and r.m.s deviation:[6]

$$\epsilon_{stat}(M,\Delta) \sim M^{-1/2}\; var^{1/2}\; \left\{\phi^{(j)}\right\} \tag{14}$$

depending (weakly) on Δ, as well as (strongly) on M. The confidence interval of the estimated potential is thus:

$$\delta(M,\Delta) = \mu_1(M,\Delta) \pm \alpha[\mu_2(M,\Delta) - \mu_1^2(M,\Delta)]^{1/2}, \tag{15}$$

where α depends on the sought confidence level.

5 COMPARISON WITH OTHER METHODS AND CONCLUSIONS

The main merits and drawbacks of the above Wiener-integral+Monte Carlo-method (henceforth WIMC) as compared to more usual methods, including finite-elements and the method of moments (henceforth FEM and MoM, respectively) for solving (4) can be summarized as follows:

- WIMC does *not* require any prior mesh-algorithm, nor is restricted to compute the potential at predetermined grid points; it can be used to compute the potential only where needed;
- WIMC has extremely mild memory requirements. In contrast, MoM and FEM require the storage of *large* matrices;
- WIMC is intrinsecally parallelizzable;
- WIMC has a rather slow (statistical, $\propto M^{-1/2}$) convergency rate. By comparison with other methods, however, it is not significantly slower. Tight accuracy bounds are also easily obtained.

As a conclusion, WIMC should be seriously considered for complicated geometries, whenever fast (possibly parallel) computing engines and relatively little memory are at hand.

The above formalism has been applied to compute the Laslett coefficients for the proposed stadium-shaped LHC liner in [7]. More or less obvious generalizations, include computation of the coherent Laslett coefficients, dielectric insertions and more general boundary conditions. This work has been sponsored in part by INFN through the Salerno University group.

6 REFERENCES

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