GENERAL ANALYSIS OF THE LONGITUDINAL COUPLING IMPEDANCE OF A RECTANGULAR SLOT IN A THIN COAXIAL LINER

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Abstract

As previously discussed [Fedotov and Gluckstern, Phys. Rev. E **54**, 1930 (1996)], for a long narrow slot whose length may be comparable with the wavelength, the usual static approximation for the polarizability and susceptibility which enter into the impedance is a poor one. Therefore, finding semi-analytic expressions for the impedance of a rectangular slot in a broad frequency range is highly desirable. We develop a general analysis based on a variational formulation, which includes both the realistic coaxial structure of the beam-pipe and the effect of finite wavelength in the calculation of the coupling impedance of a rectangular slot in a liner wall of zero thickness. We then present a numerical study of the frequency dependence of the coupling impedance of a transverse rectangular slot.

1 INTRODUCTION

We have earlier considered a liner with a symmetric annular slot.[1] This time we consider the azimuthally asymmetric problem of a single rectangular slot in an inner conductor (liner) of zero thickness.

We call the region inside the inner conductor $r \leq a$ the "pipe region" and the region outside the inner conductor $a \leq r \leq b$ the "coaxial region". The technique consists of expanding fields in both regions into a complete set of functions. At the common interface the fields have to be matched, yielding equations for the expansion coefficients. The solution is then obtained by finally truncating and inverting the resulting matrix equations.

The longitudinal coupling impedance can be written in the following form

$$\frac{Z_{\parallel}(k)}{Z_0} = -\frac{1}{2\pi a Z_0 I_0} \int dS E_z(a,\theta,z;k) e^{jkz}, \quad (1)$$

where the surface integral is only over the hole, since E_z vanishes on the liner wall. Since the driving current on axis is proportional to $\exp(-jkz)$, the problem is simplified by obtaining $Z_{\parallel}(k)$ for an even driving current $\cos kz$ and an odd driving current $-j\sin kz$ separately and then taking their sum. We should note that the variational method becomes possible only when the problem is separated into an even and an odd part. In the even problem E_z, H_r, H_θ are even in z, while in the odd problem E_z, H_r, H_θ are odd in z (where the center of the hole is chosen to be at $z = 0$, $\theta = 0$). In any case E_z, E_r, H_θ are always even in θ, and H_z, H_r, E_θ are always odd in θ. We use the superscript (e) for the even problem and the superscript (o) for the odd problem.

2 THE EVEN PART OF IMPEDANCE

Due to the asymmetry of the problem we now have θ dependence, and therefore need to use both TM and TE modes. For the TM portion of the modes we have

$$E_z^{(e)}(r,\theta,z) = \int dq \cos qz \phi^{(e)}(r,\theta), \quad (2)$$

where

$$\phi^{(e)}(r,\theta) = \sum_n \cos n\theta A_n^{(e)}(q) \left[\frac{J_n(\kappa r)}{J_n(\kappa a)}, \frac{F_n(\kappa r)}{F_n(\kappa a)}\right]. \quad (3)$$

Here we use the notation where the first part in square brackets corresponds to the pipe region $r \leq a$, and the second part corresponds to the coaxial region $a \leq r \leq b$, where F_n is the solution of the Maxwell equations for the coaxial region for the TM modes $[F_n(u) = Y_n(u)J_n(\kappa b) - J_n(u)Y_n(\kappa b)]$. The other TM field components can be obtained using Eq. (2).

For the TE portion of the modes we similarly have

$$Z_0 H_z^{(e)}(r,\theta,z) = \int dq \sin qz \psi^{(e)}(r,\theta), \quad (4)$$

where

$$\psi^{(e)}(r,\theta) = -\sum_n \sin n\theta B_n^{(e)} \left[\frac{J_n(\kappa r)}{J_n'(\kappa a)}, \frac{G_n(\kappa r)}{G_n'(\kappa a)}\right], \quad (5)$$

where $G_n(\kappa r)$ is the solution of the Maxwell equations for the coaxial region for the TE modes $[G_n(u) = Y_n(u)J_n'(\kappa b) - J_n(u)Y_n'(\kappa b)]$. The other TE field components can be obtained using Eq. (4).

For the general solution, we must include both TM and TE modes. The continuity of $H_z^{(e)}$ and $H_\theta^{(e)}$ in the hole at $r = a$ leads to

$$\int dS' E_{\theta'}^{(e)}(a,\theta',z') K_{22}^{(e)} + \int dS' E_{z'}^{(e)}(a,\theta',z') K_{21}^{(e)} = 0, \quad (6)$$

$$\int dS' E_{z'}^{(e)}(a,\theta',z') K_{11}^{(e)} + \int dS' E_{\theta'}^{(e)}(a,\theta',z') K_{12}^{(e)} = a Z_0 I_0 2\pi \cos kz, \quad (7)$$

where

$$K_{11}^{(e)} = a\sum_n \int dq \cos qz \cos qz' \cos n\theta \cos n\theta' k_{11}, \quad (8)$$

$$K_{12}^{(e)} = K_{21}^{(e)} = a\sum_n \int dq \cos qz \sin qz' \sin n\theta' \cos n\theta k_{12}, \quad (9)$$

$$K_{22}^{(e)} = a\sum_n \int dq \sin qz \sin qz' \sin n\theta \sin n\theta' k_{22}, \quad (10)$$

with

$$k_{11} = jkaP_n(q) - j\frac{q^2n^2}{\kappa^2 ka}Q_n(q), \quad (11)$$

$$k_{12} = k_{21} = j\frac{qn}{k}Q_n(q), \quad (12)$$

$$k_{22} = -j\frac{\kappa^2 a}{k}Q_n(q). \quad (13)$$

Here the functions $P_n(q)$ and $Q_n(q)$ are given by the following expressions

$$P_n(q) = \left[\frac{J_n'(\kappa a)}{\kappa a J_n(\kappa a)} - \frac{F_n'(\kappa a)}{\kappa a F_n(\kappa a)}\right], \quad (14)$$

$$Q_n(q) = \left[\frac{J_n(\kappa a)}{\kappa a J_n'(\kappa a)} - \frac{G_n(\kappa a)}{\kappa a G_n'(\kappa a)}\right]. \quad (15)$$

We now treat P_n and Q_n as functions of κa with $\kappa b = (b/a)\kappa a$ and express these functions as a sum over the zeros of the respective denominators. The detailed expansion of functions P_n and Q_n in terms of algebraic series is given in [2]. The resulting expressions for $K_{11}^{(e)}$, $K_{12}^{(e)}$, $K_{21}^{(e)}$ and $K_{22}^{(e)}$, in Eqs. (8)-(10), can then be integrated over q by means of the residue theorem.

From Eq. (1) the even part of the impedance is

$$\frac{Z_\parallel^{(e)}}{Z_0} = \frac{-1}{2\pi a Z_0 I_0}\int dS E_z^{(e)}(a,\theta,z)\cos kz. \quad (16)$$

Using Eqs. (6) and (7) we can form

$$\begin{aligned} Z_0/Z_\parallel^{(e)} = \\ - & \left(\int\int dS' dS[2E_{\theta'}^{(e)}(a,\theta',z')E_z^{(e)}(a,\theta,z)K_{12}^{(e)}\right. \\ + & E_{z'}^{(e)}(a,\theta',z')E_z^{(e)}(a,\theta,z)K_{11}^{(e)} \\ + & \left. E_{\theta'}^{(e)}(a,\theta',z')E_\theta^{(e)}(a,\theta,z)K_{22}^{(e)}]\right) \\ / & \left(\int dS E_z^{(e)}(a,\theta,z)\cos kz\right)^2, \end{aligned} \quad (17)$$

a form independent of the normalization of the fields. If we ask that the numerator of Eq. (17) be a minimum with respect to the variations of $E_z^{(e)}$ and $E_\theta^{(e)}$, subject to the constraint $\int dS E_z^{(e)}(a,\theta,z)\cos kz = 1$, we confirm that Eq. (17) is a variational form for the impedance.

After evaluation of the integrals, we obtain the final form for the even part of the impedance.[2]

3 THE ODD PART OF IMPEDANCE

We now consider the portion of the problem where $E_z^{(o)}$ is odd in z. We perform field expansions similar to those for the even portion of the problem. The continuity of $H_z^{(o)}$ and $H_\theta^{(o)}$ in the hole at $r = a$ leads to the following integral equations

$$\int dS' E_{\theta'}^{(o)}(a,\theta',z')K_{22}^{(o)} + \int dS' E_{z'}^{(o)}(a,\theta',z')K_{21}^{(o)} = 0, \quad (18)$$

$$\int dS' E_{z'}^{(o)}(a,\theta',z')K_{11}^{(o)} + \int dS' E_{\theta'}^{(o)}(a,\theta',z')K_{12}^{(o)} = aZ_0I_0 2\pi(-j\sin kz), \quad (19)$$

where

$$K_{11}^{(o)} = a\sum_n \int dq \sin qz \sin qz' \cos n\theta \cos n\theta' k_{11}, \quad (20)$$

$$K_{12}^{(o)} = K_{21}^{(o)} = -a\sum_n \int dq \sin qz \cos qz' \cos n\theta \sin n\theta' k_{12}, \quad (21)$$

$$K_{22}^{(o)} = a\sum_n \int dq \cos qz \cos qz' \sin n\theta \sin n\theta' k_{22}, \quad (22)$$

with k_{ij} given by Eqs. (11)-(13).

From Eq. (1) the odd part of the impedance is

$$\frac{Z_\parallel^{(o)}}{Z_0} = \frac{-j}{2\pi a Z_0 I_0}\int dS E_z^{(o)}(a,\theta,z)\sin kz. \quad (23)$$

Using Eqs. (18) and (19) we can form

$$\begin{aligned} Z_0/Z_\parallel^{(o)} = \\ - & \left(\int\int dS' dS[2E_{\theta'}^{(o)}(a,\theta',z')E_z^{(o)}(a,\theta,z)K_{12}^{(o)}\right. \\ + & E_{z'}^{(o)}(a,\theta',z')E_z^{(o)}(a,\theta,z)K_{11}^{(o)} \\ + & \left. E_{\theta'}^{(o)}(a,\theta',z')E_\theta^{(o)}(a,\theta,z)K_{22}^{(o)}]\right) \\ / & \left(\int dS E_z^{(o)}(a,\theta,z)\sin kz\right)^2, \end{aligned} \quad (24)$$

which is the variational form for the odd part of the impedance. After evaluation of the integrals, we obtain the final form for calculation of the odd part of the impedance [2].

4 NUMERICAL RESULTS AND DISCUSSIONS

Formulas for direct numerical computation of the impedance of a rectangular slot have been obtained.

As an example, and to test our formulas, we present here a numerical study of the impedance of rectangular slots of different azimuthal length. In order to compare our results with those presented by Filtz and Scholz[3] we also choose the parameters of the LHC design.

In Figs. 1 and 2, the frequency dependence of both the real and imaginary parts of the coupling impedance of the transverse rectangular slot is presented, with an angular length 180 and 350 degrees, respectively. As expected, the behavior with respect to frequency strongly differs from the one of slots with the small angular length, even for relatively low frequencies.

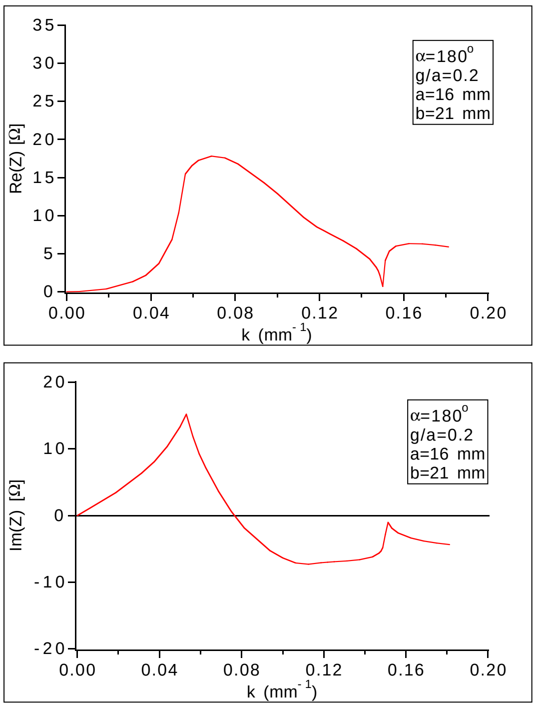

Figure 1: Real and imaginary parts of the coupling impedance of transverse rectangular slot with the azimuthal length 180 degrees.

The general behavior of the impedance agrees with the results of Filtz and Scholz, even though there is some shift between our results and theirs. They used a similar field matching technique, and their calculations of impedance were apparently performed without using a variational form, presumably requiring large matrices and considerable CPU time. Our approach, which uses a variational form, requires only modest size matrices.

5 SUMMARY

The main purpose of this paper is to present a detailed analysis of the calculation of the coupling impedance of a rectangular slot in a coaxial liner of negligible wall thickness over a wide frequency range. We obtain equations for calculating the even and odd parts of the impedance, expressed in variational form. The use of the variational

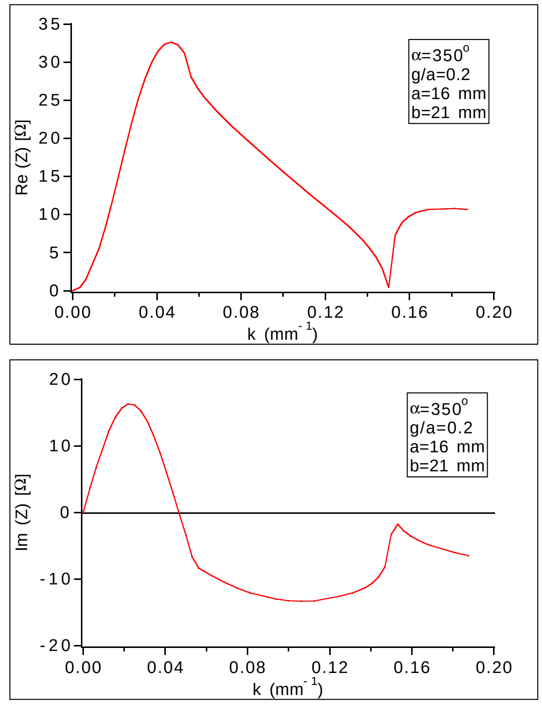

Figure 2: Real and imaginary parts of the coupling impedance of transverse rectangular slot with the azimuthal length 350 degrees.

method makes numerical study fast and accurate. The numerical results obtained for both the imaginary and the real part correspond to the expected ones for frequencies below and above cutoff.

6 ACKNOWLEDGMENTS

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7 REFERENCES

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