BEAM LOADING OPTIMIZED FILLING PATTERNS FOR PARTIALLY FILLED STORAGE RINGS

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Abstract

A method of selecting a favorable, with respect of the beam loading effects, multi-bunch (M bunches) filling pattern for a partially filled ring (harmonic number N) is presented. Starting with a standard set of wake field-coupled equations of motion, the beam loading effects (higher order mode loss) are evaluated analytically (using contour integration technique) for a partially filled ring (M < N). Resulting analytic formula describe the mode loss experienced by a given bunch within a train, as a function of the resonant frequency and the quality factor of the coupling impedance, Q. The analytic formula reveals filling pattern dependent resonant frequency regions, where the beam loading effects are highly suppressed. A possible application of the presented formalism is that for a given configuration of cavity resonances one can design the optimum bunch filling pattern.

1 INTRODUCTION

While coupled multi-bunch motion for a symmetric configuration of populated buckets in a storage ring has been extensively studied and the stability problem has a closed analytic solution [1] for most standard wake fields a fully populated ring is rarely the case for any operational mode of a realistic synchrotron.

We present a rigorous treatment of the beam loading effects for a non symmetric configuration of populated buckets. The core of this paper is an analytic method, involving contour integration in complex frequency domain, which yields a closed expressions describing the mode loss due to a general resonant impedance, for the case of partially filled ring.

The parasitic mode loss is expressed explicitly in of: the bunch index, the resonance frequency and the quality factor of the impedance peak. Superimposing many parasitic cavity modes one can use the above formulas to choose appropriate tuning of existing configuration of parasitic modes to minimize mode loss effects, or to optimize the filling bunch pattern (number of bunches in a train) to suppress the beam loading effects.

2 COUPLED MULTI-BUNCH MOTION

We assume a storage ring of a harmonic number N populated by a train of M consecutive bunches ($M \le N$). For the dipole mode consideration, it will be sufficient to model each bunch as a macro particle combining intensity of the entire bunch. To describe a coupled motion of a system of M bunches we represent a state of the system at a given time by a vector. Its n-th component describes the longitudinal coordinate of the n-th bunch (y_n is the phase of the bunch with respect to the center of an unperturbed bucket).

Collective synchrotron motion of the system on M bunches coupled via wake fields can be described by the following set of equations of motion [1].

$$\frac{\partial^2}{\partial t^2} y_n(t) + \omega^2 y_n(t) = A \sum_{k=-\infty}^{\infty} \sum_{m=0}^{M-1} \left\{ W'\left(-(k + \tfrac{m-n}{N})T_0\right) + \right. \tag{1}$$

$$\left. \frac{1}{c} W''\left(-(k+\tfrac{m-n}{N})T_0\right)\left[y_n(t) - y_m\left(t - (k + \tfrac{m-n}{N})T_0\right)\right]\right\}$$

$$A = \frac{N_0 r_0 \eta \omega_0}{2\pi\gamma} .$$

Here W' and W'' are the time derivatives of the wake function, ω is the unperturbed synchrotron frequency, η is the phase slip factor, c is the velocity of light, r_0 is the classical proton radius, ω_0 is the revolution frequency and T_0 is the revolution period. The index k gives the sum of the wake fields from all previous turns.

One can identify the first term in the right hand side of Eq.(1) with the mode loss [2]–[3] suffered by the n-th bunch. It explicitly depends on the bunch index n. One can notice in passing, that the mode loss can be viewed as a shift of the minimum of the potential well − the synchronous phase shift − f_n.

$$f_n = A \sum_{k=-\infty}^{\infty} \sum_{m=0}^{M-1} W'\left(-(k + \tfrac{m-n}{N})T_0\right) . \tag{2}$$

The term proportional to y_n in the right hand side of Eq.(1) can be absorbed by the synchrotron frequency. This is known as the synchrotron frequency shift, $\Delta\omega_n$, due to the potential well distortion [3]–[4] (change of its curvature) and it is given by the following formula

$$\Delta\omega_n^2 = A\,\frac{1}{c} \sum_{k=-\infty}^{\infty} \sum_{m=0}^{M-1} W''\left(-(k + \tfrac{m-n}{N})T_0\right) . \tag{3}$$

The resulting set of equations, Eqs.(1)–(3), along with a convenient representation of the wake field coupling, will be analyzed in the complex frequency domain later in the paper.

3 PARASITIC MODE LOSS − PARTIALLY FILLED RING

To evaluate the mode loss term (or the incoherent synchrotron tune shift) it is convenient to express it in terms of the longitudinal coupling impedance via the inverse Fourier transform. The resulting expression, Eq(2) is rewritten as follows

$$f_n = \frac{Ac}{2\pi} \sum_{k=-\infty}^{\infty} \sum_{m=0}^{M-1} \int_{-\infty}^{\infty} d\omega \, e^{-i\omega(k + \frac{m-n}{N})T_o} Z_{||}(\omega) . \qquad (4)$$

Infinite summation over k can be carried out explicitly using a trivial version the Poisson sum identity. This yields the following expression

$$f_n = \omega_0 \frac{Ac}{2\pi} \sum_{p=-\infty}^{\infty} \sum_{m=0}^{M-1} Z_{||}(p\omega_0) \; e^{2\pi i p \frac{n-m}{N}} . \qquad (5)$$

Applying a simple sum identity to Eq.(5) one can rewrite it in the following form

$$f_n = \omega_0 \frac{Ac}{2\pi} \sum_{q=-\infty}^{\infty} \sum_{l=0}^{N-1} e^{2\pi i l \frac{n}{N}} Z_{||}\big((Nq+l)\omega_0\big) \sum_{m=0}^{M-1} e^{-2\pi i l \frac{m}{N}} . \qquad (6)$$

the last summation (over m) in Eq.(6) can be carried out explicitly. The resulting formula is written as follows

$$f_n = \omega_0 \frac{Ac}{2\pi} \sum_{p=-\infty}^{\infty} e^{2\pi i (p\omega_o)\frac{n}{N\omega_o}} Z_{||}(p\omega_0) \times \qquad (7)$$

$$\frac{1 - e^{-2\pi i (p\omega_o)\frac{M}{N\omega_o}}}{1 - e^{-2\pi i (p\omega_o)\frac{1}{N\omega_o}}} = \sum_{p=-\infty}^{\infty} F(p\omega_0) \quad .$$

The second equality in Eq.(7) highlights generic sampling structure of the above expression. Applying the Poisson sum identity to Eq.(7) yields the following expression

$$f_n = iAc \sum_{q=-\infty}^{\infty} \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega \, Z_{||}(\omega) \frac{e^{2\pi i (q + \frac{n}{N}) \frac{\omega}{\omega_o}} - e^{2\pi i (q - \frac{M-n}{N}) \frac{\omega}{\omega_o}}}{1 - e^{-2\pi i \frac{\omega}{N\omega_o}}} . \qquad (8)$$

Introducing two kinds of generic integrals, namely:

$$I^{+}(k) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega \; Z_{||}(\omega) \; \frac{e^{2\pi i k \frac{\omega}{N\omega_o}}}{1 - e^{-2\pi i \frac{\omega}{N\omega_o}}} \; , \qquad k \geq 0$$

$$I^{-}(k) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega \; Z_{||}(\omega) \; \frac{e^{-2\pi i k \frac{\omega}{N\omega_o}}}{1 - e^{-2\pi i \frac{\omega}{N\omega_o}}} \; , \qquad k \geq 0 \qquad (9)$$

one can express the mode loss term, Eq.(8), in the following compact form

$$f_n = iAc \Big\{ I^{+}(n) - I^{-}(k) + \sum_{q=1}^{\infty} \Big[I^{+}(Nq+n) + I^{-}(Nq-n) + \qquad (10)$$

$$- I^{+}\big(Nq - (M-n)\big) - I^{-}\big(Nq + (M-n)\big) \Big] \Big\} \; .$$

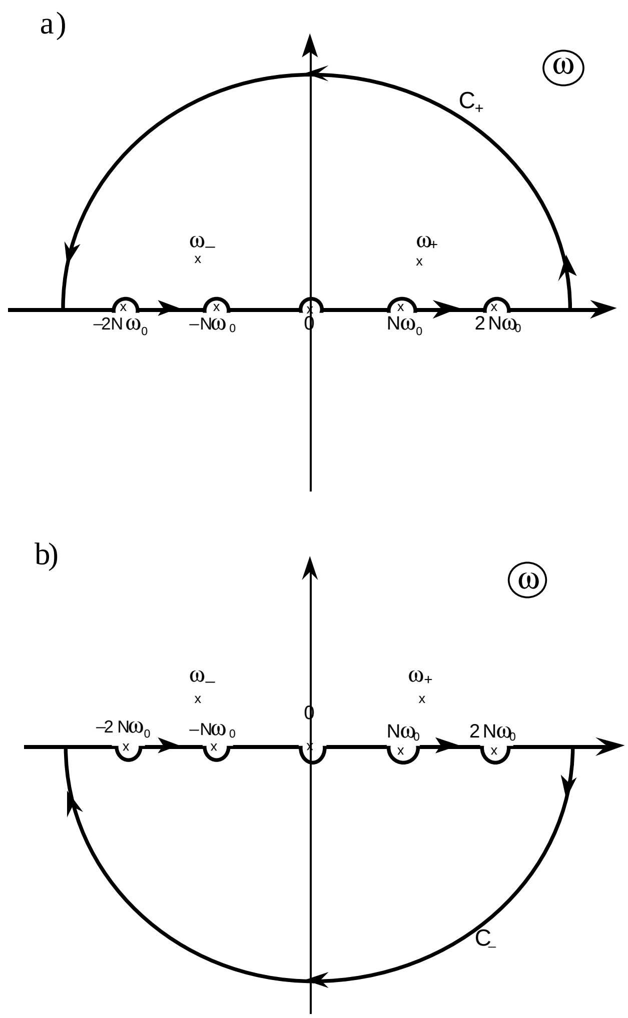

Figure: 1 Complete set of singularities along with the appropriate choice of integration contours for both $I^{+}(k)$ and $I^{-}(k)$ – C_+ and $C\_$ respectively.

Assuming general form of the longitudinal impedance of a resonant structure, given by the following standard expression:

$$Z_{||}(\omega) = \frac{R}{1 + iQ\left(\frac{\omega}{\omega_r} - \frac{\omega_r}{\omega}\right)} = -\, iR \, \frac{\omega_r}{Q} \, \frac{\omega}{(\omega - \omega_+)(\omega - \omega_-)} \; , \qquad (11)$$

where R is the shunt resistance, Q is the quality factor of the resonator and ω_r is its resonance frequency, the singularities of $Z_{||}(\omega)$ are defined by the following pair of complex poles $\omega_{\pm}$ located in the upper half plane

$$\omega_\pm = \omega_r(\pm 1 + i\delta), \qquad \delta = \frac{1}{2Q} << 1 \quad . \tag{12}$$

The integrals, $I^\pm$, can be easily evaluated through contour integration via Cauchy's integral theorem. The choice if the integration contours is illustrated in Figure 1.

After evaluating the integrals, $I^\pm$, explicitly [4], one can rewrite Eq.(10) in the following closed form

$$\begin{aligned} f_n = \frac{AcR\omega_o}{\pi}\Bigg\{ & \left(\frac{\pi\omega_r}{N\omega_o}\right)\delta^2 \frac{M\left(\frac{\pi\omega_r}{N\omega_o}\right) - \frac{1}{2}N\sin\left(\frac{2\pi\omega_r}{N\omega_o}\right)}{\sin^2\left(\frac{\pi\omega_r}{N\omega_o}\right) + \left(\frac{\pi\omega_r}{N\omega_o}\right)^2\delta^2} + \\ & - \frac{\sin\left(\frac{\pi\omega_r}{N\omega_o}\right)\sin\left(\frac{\pi\omega_r}{N\omega_o}M\right)\cos\left(\frac{\pi\omega_r}{N\omega_o}(2n-M-1)\right)}{\sin^2\left(\frac{\pi\omega_r}{N\omega_o}\right) + \left(\frac{\pi\omega_r}{N\omega_o}\right)^2\delta^2}\Bigg\}. \end{aligned} \tag{13}$$

Denoting the expression in curly bracket by $\tilde{f}_n$, one can introduce a dimensionless mode loss. Figure 2 illustrates a family of curves for different values of n, calculated according to Eq.(13).

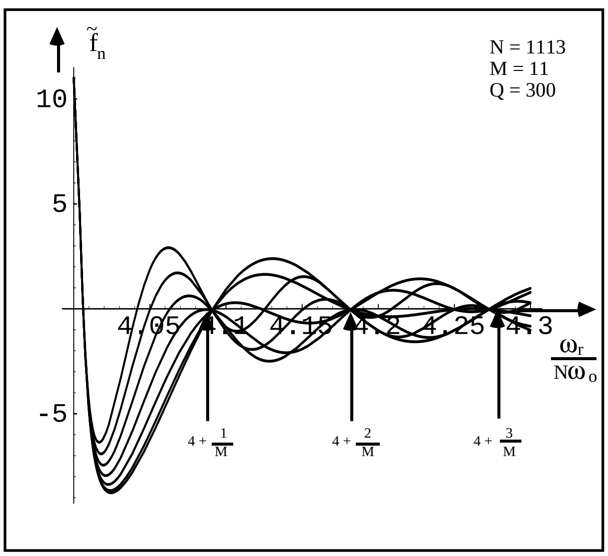

Figure: 2 Dimensionless mode loss, $\tilde{f}_n$, suffered by the n-th bunch – a discrete set of resonant frequencies, ω_r, defined by the fractional, $1/M$, multiples of $N\omega_o$.

As one can see Eq.(13) has a simple asymptotics for the resonance frequencies, ω_r, in the vicinity of the integer multiples of the r.f. frequency, $kN\omega_o$, and away from them. These two asymptotic regions are determined by the relative strength of the expressions appearing in the denominator of Eq.(13), namely: $\sin^2(\pi x)$ and $(\pi x)^2\delta^2$. One can notice, that for the resonance frequencies in 'the immediate vicinity of the integer multiple of the r.f. frequency' the resulting mode loss does not depend on the bunch index, n, and it is governed by the quality factor, Q. Conversely, for the resonance frequencies outside that region, the resulting mode loss does not depend on the quality factor, Q, and it is governed strictly by the bunch index, n, which explains why the so called 'de-Q-ing' of the modes does not have any effect on the beam loading effects for a partially filled ring.

Furthermore, the structure of Eq.(13) – zeros of $\sin(\pi xM)$ – reveal another finer level of symmetry defined by the fractional, $1/M$, multiples of $N\omega_o$. The mode loss vanishes up to terms of $O(\delta^2)$, for a discrete set of resonance frequencies marked in Figure 2 (arrows).

4 SUMMARY – OPTIMIZED FILLING PATTERN

The mode loss experienced by a given bunch within the train, were calculated analytically (using contour integration technique) for a partially filled ring. Resulting simple analytic formulas express both quantities, as a function of the resonance frequency, ω_r, and the quality factor of the coupling impedance, Q.

The formula reveals a set of characteristic resonant frequencies, spaced by the multiples of $N\omega_o/M$, at which the potential well distortion characteristics are not only bunch independent, but also considerably smaller (it scales as MQ^{-2}).

Finally, for a given configuration of cavity resonances one can get immediately a simple quantitative answer in terms of the mode loss and the synchrotron tune shift experienced by each bunch along the train [4]. These analytic expressions give one an insight into various optimizing schemes; e.g. to modify the existing configuration of parasitic cavity resonances (via frequency tuning), so that the resulting potential well distortion effects are minimized. One can also explore existence of the beam loading suppression frequencies, which depends on the bunch train index, M, to design an optimum filling pattern for a storage ring, e.g. a train of trains to maximize the net beam intensity, which is especially useful for a synchrotron light source to maximize its brightness and still avoid undesirable mode loss effects.

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