SYNCHROTRON RADIATION WAKE IN FREE SPACE

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Abstract

A transverse force acting on particles of a short bunch coherently radiated in free space is derived.

1 INTRODUCTION

In this paper, we derive the transverse radiation force of a bunch of ultrarelativistic charged particles coherently radiating in free space assuming that the bending radius is much larger than the beam dimensions. In contrast to a similar recent study [1], where the authors decompose the total transverse force and find only a part that is responsible for the distortion of the beam orbit, we derive a full expression for the force and leave the issues of the beam dynamics for a separate consideration. Another approach to the calculation of the transverse force has been previously developed in [2].

In many cases considered in this paper, the calculations are extremely cumbersome; they were systematically performed with the use of symbolic engine of the computer program MATHEMATICA [3].

2 POTENTIALS AND FIELDS

Our approach is based on Taylor expansion of the electrostatic potential φ and the vector potential $\mathbf{A}$ in the vicinity of the particle. For a moving point charge, the potentials are given by the Lienard-Wiechert formula [4]

$$\varphi(\mathbf{r},t) = \frac{e}{(R-\boldsymbol{\beta}_{\rm ret}\mathbf{R})},\ \mathbf{A}(r,t) = \frac{e\boldsymbol{\beta}_{\rm ret}}{(R-\boldsymbol{\beta}_{\rm ret}\mathbf{R})}, \quad (1)$$

where $\mathbf{r}$ and t refer to the observation point and time, respectively, e is the particle charge, $\beta_{\rm ret}$ is the ratio $\mathbf{v}/c$ evaluated at the retarded time $t_{\rm ret}$, and $\mathbf{R} = \mathbf{r} - \mathbf{r}_{\rm ret}$, where $\mathbf{r}_{\rm ret}$ gives the location of the particle at the retarded time. The retarded time is determined by the equation $R = c(t - t_{\rm ret})$. The particle is considered to be extremely relativistic, so that the parameter $\delta \equiv \gamma^{-2} = 1 - \beta^2 \ll 1$.

We assume that the particle moves along a circular orbit of radius ρ, as shown in Fig. 1. The vector $\mathbf{r}_{\rm p}$ refers to the position of the particle at time t. We introduce the angle α between the vectors $\mathbf{r}_{\rm ret}$ and $\mathbf{r}_{\rm p}$, and the angle ψ between the vector $\mathbf{r}_{\rm p}$ and the projection of the vector $\mathbf{r}$ onto the plane of motion. The causality principle requires that $\alpha > 0$, i.e. the radiation observed at t is emitted by the particle prior to the observation time.

In a polar coordinate system $(r,\ \theta,\ y)$ with the origin located at the center of the orbit, the radius-vector $\mathbf{r}$ of the observation point is represented by $\mathbf{r} = (\rho + x, \theta_p + \psi, y)$, where $\theta_p = c\beta t/\rho$ is the angle corresponding to the position of the particle at time t. In this coordinate system, the equation $R^2 = c^2(t - t_{ret})^2$ which determines the angle α can be written as

$$\frac{\alpha^2}{\beta^2} = 1 + \left(1+\frac{x}{\rho}\right)^2 + \frac{y^2}{\rho^2} - 2\left(1+\frac{x}{\rho}\right)\cos(\psi+\alpha). \quad (2)$$

In terms of α, ψ and x, the denominator in Eq. (1) is

$$R - \boldsymbol{\beta}_{\rm ret}\mathbf{R} = \rho\frac{\alpha}{\beta} - \beta(\rho + x)\sin(\psi+\alpha), \quad (3)$$

and the polar components of the vector potential are

$$A_r = \frac{e\beta\sin(\alpha+\psi)}{R-\boldsymbol{\beta}_{\rm ret}\mathbf{R}},\quad A_\theta = \frac{e\beta\cos(\alpha+\psi)}{R-\boldsymbol{\beta}_{\rm ret}\mathbf{R}}. \quad (4)$$

We expand the potentials in the Taylor series assuming

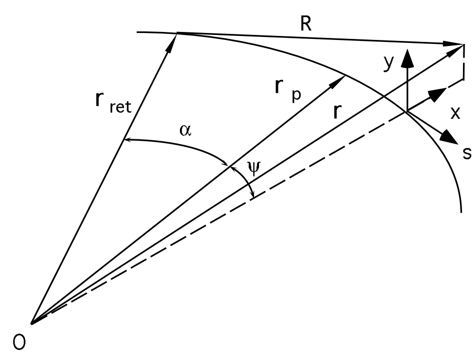

Figure 1: Particle trajectory and coordinate system.

that $\psi \ll 1$ and $x,\ y \ll \rho$. In order to perform Taylor expansions, we need to solve Eq. (2) for α and substitute the solution into Eqs. (3), (1) and (4). For given potentials, the electric field can be found by differentiating φ and $\mathbf{A}$:

$$\mathbf{E} = -\nabla\varphi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} = -\nabla\varphi + \frac{\beta}{\rho}\frac{\partial\mathbf{A}}{\partial\psi},\ \mathbf{H} = \mathbf{R}\times\frac{\mathbf{E}}{R}. \quad (5)$$

After finding $\mathbf{E}$ and $\mathbf{H}$, we calculate the Lorentz force $\mathbf{F}$ acting on a test particle located at point $\mathbf{r}$ such that $|\mathbf{r} - \mathbf{r}_{\rm p}| \ll \rho$, and moving with the velocity $\mathbf{v} = c\boldsymbol{\beta}$ along the circle in the same direction as the driving particle, $\boldsymbol{\beta} = (0, \beta, 0)$. Integration of this force with a Gaussian distribution function gives an expression for the radiation force acting on a Gaussian bunch due to the emission of the radiation in free space.

3 DERIVATION OF TRANSVERSE FORCE FOR A POINT CHARGE

For the sake of simplicity, we assume that the test particle trajectory lies in the plane $y = 0$. In addition, we also neglect the difference between the particle velocity and the speed of light and set $\delta = 0$.

In order to keep track of relative orders of different variables, it is convenient to assign to each variable a formal small parameter ε which will be set to unity in the final result. We choose ordering such that $\alpha \sim \varepsilon$, $\psi \sim \varepsilon^3$, $x/\rho \sim \varepsilon^2$, where ε is a small formal parameter. Expanding Eq. (2) and keeping terms up to fourth order of ε, yields a quartic equation for α,

$$-\frac{\alpha^4}{12} + 2\alpha\psi + \alpha^2\frac{x}{\rho} + \left(\frac{x}{\rho}\right)^2 = 0. \tag{6}$$

The real positive solution α to this equation as a function of ψ and x can be written in terms of a parametric dependence,

$$\alpha = \sqrt{q} f\left(\frac{x}{\rho q}\right), \quad \psi = q^{3/2} g\left(\frac{x}{\rho q}\right), \tag{7}$$

where q is a positive parameter varying from 0 to ∞, and the functions $f(\xi)$ and $g(\xi)$ are

$$\begin{aligned} f(\xi) &= \pm 1 + \left(-1 + 6\xi + 2\sqrt{1 - 6\xi + 12\xi^2}\right)^{1/2}, \\ g(\xi) &= \pm\frac{1}{3}\sqrt{1 - 6\xi + 12\xi^2}. \end{aligned} \tag{8}$$

The upper and lower signs in Eq. (8) correspond to the positive and negative values of ψ respectively. From Eq. (7) it follows that the product $\alpha\,|\rho/x|^{1/2}$ depends on the variable $\psi\,|\rho/x|^{3/2}$ only. In the limit $\psi \gg (x/\rho)^{3/2}$, approximately $\alpha \approx 2(3\psi)^{1/3}$. For large negative values of ψ, $|\psi| \gg (x/\rho)^{3/2}$, we find $\alpha \approx -x^2/2\psi\rho^2$. However, as a detailed analysis shows, the applicability of Eqs. (7) for negative ψ is limited by the condition $|x| \gg |\psi|$. Larger absolute values of (negative) ψ are considered below.

An important feature of our solution which follows from Eqs. (7) and (8) is that for each positive value of x the derivative of the function $\partial\alpha/\partial\psi$ becomes infinite at some point $\psi = \psi_0$. A simple calculation yields for ψ_0,

$$\psi_0 = -\frac{1}{3}\left(2\frac{x}{\rho}\right)^{3/2}, \quad \alpha(\psi_0, x) = \sqrt{2x}. \tag{9}$$

When x varies from 0 to ∞, the first of the equations (9) determines a curve in the $(x,\ \psi)$ plane which we will call a *singular line*, because both potentials and the fields will have a singularity here. It is important to emphasize that the singularity occurs only because we assume $\delta = 0$; for finite, though small δ, the fields would be limited everywhere as soon as $x \neq 0$. Note also that the parameter q on the singular line equals $2x$.

The origin of the singular line can be understood from the following geometrical consideration. If we draw wavefronts of the radiation emitted at different times by a particle moving with the speed of light around a circle, they will form a pattern shown schematically in Fig. 2. The wavefronts are condensed on the outer side of the circle forming a caustic which, as one can show, coincides with the singular line found above and shown in Fig. 2 by the dashed curve. The superposition of multiple wavefronts on the caustic in the limit $v = c$ gives rise to the infinitely large fields on the singular line.

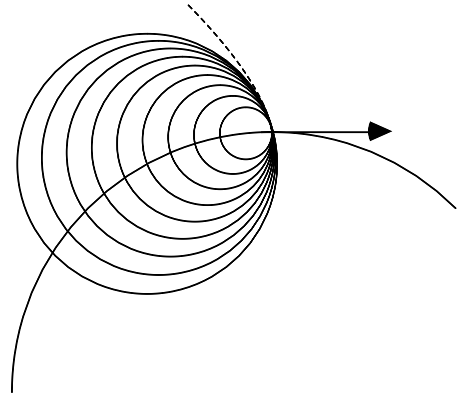

Figure 2: Formation of a singular line (dashed curve) due to radiation of ultrarelativistic particle moving around a circle. The circles show wavefronts emitted by the particle at different times.

Expanding Eqs. (3) and (1) we find for the potential φ

$$\varphi = \frac{e}{\rho}\left(\frac{\alpha^3}{6} - \psi - \alpha\frac{x}{\rho}\right)^{-1}, \tag{10}$$

and for the components of the vector potential

$$\begin{aligned} A_r &= \frac{e}{\rho}\alpha\left(\frac{\alpha^3}{6} - \psi - \alpha\frac{x}{\rho}\right)^{-1}, \\ A_\theta &= \frac{e}{\rho}\left(1 - \frac{\alpha^2}{2}\right)\left(\frac{\alpha^3}{6} - \psi - \alpha\frac{x}{\rho}\right)^{-1}. \end{aligned} \tag{11}$$

The longitudinal electric field can be calculated using the formula

$$E_\theta = \frac{1}{\rho}\frac{\partial A_\theta}{\partial\psi} - \frac{1}{(\rho + x)}\frac{\partial\varphi}{\partial\psi} \approx \frac{x}{\rho^2}\frac{\partial\varphi}{\partial\psi} - \frac{1}{\rho}\frac{\partial(\varphi - A_\theta)}{\partial\psi}, \tag{12}$$

and for the transverse electric field we have

$$E_r = -\frac{\partial\varphi}{\partial x} + \frac{1}{\rho}\frac{\partial A_r}{\partial\psi}. \tag{13}$$

For the radial component of the transverse force we have

$$\begin{aligned} F_r &= E_r + (\boldsymbol{\beta} \times \mathbf{H})_r \\ &= E_r\left(1 - \beta\cos\left(\frac{\alpha + \psi}{2}\right)\right) + \frac{1}{2}E_\theta\sin\left(\frac{\alpha + \psi}{2}\right) \\ &\approx \frac{1}{8}\left(\alpha + \frac{2x}{\rho\alpha}\right)^2 E_r + \frac{1}{2}\left(\alpha + \frac{2x}{\rho\alpha}\right)E_\theta. \end{aligned} \tag{14}$$

After some algebra F_r can be written as

$$F_r = \frac{e}{\rho^2}\,|\rho/x|^{3/2}\,G\left(\psi\,|x/\rho|^{3/2}\right), \tag{15}$$

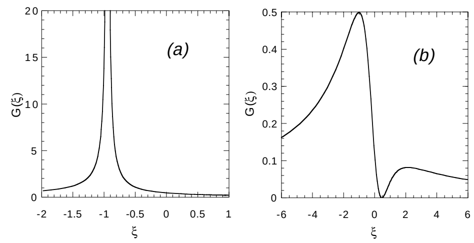

Figure 3: Function $G(\xi)$ for positive (*a*) and negative (*b*) values of x. For positive x, this function has a singularity at $\xi = -2^{3/2}/3 = -0.94$.

where the function G is different for positive and negative x. Plots of the function $G(\xi)$ are shown in Fig. 3. It is easy to check that the denominator in Eqs. (10) and (11) vanishes on the singular line and causes the potentials to diverge. Expansions of φ and A for small values of $|\psi - \psi_0|$ give the following singularity for F_r in the vicinity of the singular line:

$$F_r \approx \frac{e}{\rho^2} \frac{2^{1/6} x^{1/2}}{3^{4/3} (\psi - \psi_0)^{4/3}}. \tag{16}$$

To find the fields behind the particle ($\psi < 0$) we use the following ordering: $\alpha \sim \psi \sim x \sim \varepsilon$. Expansion of Eq. (2) up to the second order yields $2\alpha\psi + \psi^2 + x^2 = 0$ with the solution $\alpha = -\left[\psi^2 + (x/\rho)^2\right]/(2\psi)$. This solution leads to the following expressions for the potentials:

$$\begin{aligned} \varphi &= -\frac{e}{\rho\psi}, \;\; A_r = -\frac{e}{\rho\psi}\left(\frac{\psi}{2} - \frac{x^2}{2\psi\rho^2}\right), \\ A_\theta &= -\frac{e}{\rho\psi}\left(1 - \frac{\psi^2}{8} + \frac{x^2}{4\rho^2} - \frac{x^4}{8\psi^2\rho^4}\right), \end{aligned} \tag{17}$$

and for the longitudinal and radial electric fields

$$E_\theta = \frac{e}{\rho^2}\left(\frac{x}{\rho^2\psi^2} + \frac{1}{8}\right), \; E_r = -\frac{e}{2\rho^2\psi}\left(1 - \frac{x^2}{\psi^2\rho^2}\right). \tag{18}$$

For the transverse force we find

$$F_r = -\frac{e}{\rho^2} \frac{x^2\left(x^2 + 3\rho^2\psi^2\right)}{\psi\left(x^2 + \rho^2\psi^2\right)^2}. \tag{19}$$

Examination of Eqs. (18) and (19) reveals that the electric field and the force F_r tend to infinity when $\psi \to 0$. These equations, however, are only valid if $|\psi| \gg (|x|/\rho)^{3/2}$; in the opposite limit Eq. (15) should be used.

Note also that immediately behind the particle on the orbit $(x = 0)$ the longitudinal electric field is equal to $e/8\rho^2$.

We now want to calculate the transverse force f_r acting on a particle of unit charge in a bunch with a two-dimensional distribution function $g(x, s)$ normalized so that $\int g(x, s) dx ds = 1$. Here we use the longitudinal coordinate s related to ψ by $s = \rho\psi$. By definition,

$$f_r(x', s') = N \int_{-\infty}^{\infty} dx ds\, g(x' - x, s' - \rho\psi)\, F_r(x, \psi) \tag{20}$$

The integration in Eq. (20) is not a simple task for two reasons. First, F_r has a peak in the region of small ψ, $|\psi| \sim (|x|/\rho)^{3/2}$, and one needs to accurately evaluate the contribution from this peak. Second, for positive x, the integrand exhibits a singularity (see Eq. (16)) which should be handled with caution. Tracing the origin of this singularity shows that it arises from the differentiation of the potentials which have an integrable singularity $\sim |\psi - \psi_0|^{-1/3}$. This prompts the technique that allows to evaluate the integral: representing F_r in terms of potentials and performing integration by parts over s. The resulting integral converges at $\psi = \psi_0$ and can be found with the help of numerical integration. This program is accomplished in Ref. [5]. To simplify the analysis we assume that the bunch length σ_s is such that

$$\sigma_s \gg \sigma_x \tag{21}$$

The result of the integration in Eq. (20) is

$$\begin{aligned} f_r(x', s') = -N\frac{e}{\rho}\Bigg[2\int_{-\infty}^{\infty} dx g(x' - x, s') \ln\left(1.1\frac{|x|}{\rho}\right) \\ + \frac{1}{3}\int_{-\infty}^{\infty} dx \int_0^{\infty} ds \frac{\partial g(x' - x, s' - s)}{\partial s} \ln\left(\frac{s}{\rho}\right)\Bigg]. \end{aligned} \tag{22}$$

Note that for a two-dimensional Gaussian distribution,

$$g(x, s) = \frac{1}{2\pi\sigma_x\sigma_s} \exp\left(-\frac{x^2}{2\sigma_x^2} - \frac{s^2}{2\sigma_s^2}\right), \tag{23}$$

the second term in Eq. (22) reduces to a product of two one-dimensional integrals, and the force will be represented as a product of two functions one of which depends on s and the other depends on x only.

For illustration, we assume that $\sigma_x/\sigma_s = 0.1$ and $\sigma_s/\rho = 10^{-4}$, and calculate the force on the beam trajectory $x' = 0$. It turns out that, in this case, the first term in Eq. (22) dominates and $f_r(0, s)$ can be well approximated by a Gaussian:

$$f_r(x, s) = 8.4\frac{Ne}{\rho\sigma_s} \exp\left(-\frac{s^2}{2\sigma_s^2}\right). \tag{24}$$

4 ACKNOWLEDGMENTS

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