GEOMETRICAL WAKE OF A SMOOTH FLAT COLLIMATOR

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Abstract

A transverse geometrical wake generated by a beam passing through a smooth flat collimator with a gradually varying gap between the upper and lower walls is considered. Based on generalization of the approach recently developed for a smooth circular taper [1] we reduce the electromagnetic problem of the impedance calculation to the solution of two much simpler static problems – a magnetostatic and an electrostatic ones. The solution shows that in the limit of not very large frequencies, the impedance increases with the ratio h/d where h is the width and d is the distance between the collimating jaws. Numerical results are presented for the NLC Post Linac collimator.

1 INTRODUCTION

In this paper we calculate the impedance of a flat smooth collimator schematically shown in Fig. 1. The collimator extends in the y direction from $y = -h$ to $y = h$ and is bounded by perfectly conducting walls. The beam propagates in the z direction. The collimator upper and lower walls are given by the equation $x = \pm b\,(z)$, where $b\,(z)$ is a smooth function such that $b'\,(z) \ll 1$. Throughout this paper we assume that $h \gg b\,(z)$.

For a smoothly varying wall and not very high frequency such that $kb^2 < l$, where $k = \omega/c \approx \sigma_z^{-1}$, and l is the length of the collimator, the energy loss of the beam due to the radiation in the collimator is small [1]. This results in the real part of the impedance being much smaller than its imaginary part, and, in the first approximation, the real part can be neglected. In this approximation, the transverse impedance is purely imaginary and does not depend on the frequency. The latter allows us to simplify its calculation considering only the limit $\omega \to 0$ [1]. In this limit, the electric field can be found as a solution to electrostatic equations, and magnetic field satisfies magnetostatic equations with proper boundary conditions. For the electrostatic problem, one has to find the electric field of a charged wire of unit charge density stretched along the beam trajectory $x = x_0$, $y = 0$. The magnetostatic problem requires finding the magnetic field generated by the same wire carrying a unit current. After the electric and magnetic fields are found, the transverse impedance in x direction can be calculated using the following formula

$$Z_x\,(x, y, x_0) = -\frac{i}{c}\int_{-\infty}^{\infty} (E_x - H_y)\,dz. \tag{1}$$

This impedance has a dimension of Ohm and depends on the location of the driving particle x_0 and the coordinates of the test particle, x, y.

The wake w corresponding to a purely imaginary transverse impedance (1) is $w\,(x, y, x_0, z) = icZ_x\,(x, y, x_0)\,\delta\,(z)$. After the passage of the collimator the bunch will be deflected in the x direction by an angle $Nr_e\kappa/\gamma$, where N is the number of particles in the bunch, r_e is the classical electron radius (for electron/positron beam), γ is the relativistic factor, and $\kappa = -c\,\mathrm{Im}Z_x/2\sqrt{\pi}\sigma_z$.

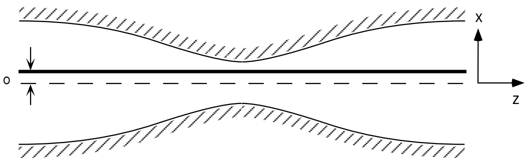

Figure 1: Sketch of a smooth flat collimator. The collimator extends from $y = -h$ to $y = h$ in the y direction. A heavy line shows the trajectory of a driving particle.

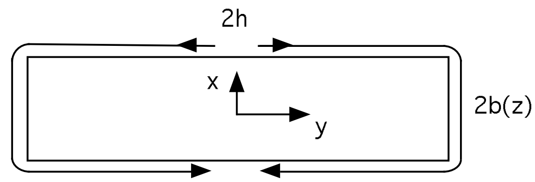

Figure 2: Cross section of the collimator by a plane $z = \mathrm{const}$ and transverse currents flowing in the collimator walls.

2 CALCULATION OF FIELDS

2.1 Electrostatic Problem

Since we assume that $h \gg b\,(z)$, for the electrostatic problem we set $h \to \infty$ and consider a collimator that extends infinitely in $\pm y$ directions. One can show that the effect of finite h in the electrostatic problem is exponentially small and can be neglected.

Let φ_e denotes the electrostatic potential such that $E_x = -\partial\varphi_e/\partial x$. It satisfies the Poisson equation with the right hand side representing a linear charge with a unit charge density,

$$\Delta\varphi_e = -4\pi\delta\,(x - x_0)\,\delta\,(y)\,, \tag{2}$$

with the boundary condition $\varphi_e|_{x=\pm b(z)} = 0$. Since the boundary $b\,(z)$ is a slow varying function of its argument, in the zero approximation, we can neglect the variation of the potential φ_e in the z direction. This assumption reduces

Eq. (2) to

$$\Delta_{x,y}\varphi_e^{(0)} = -4\pi\delta\left(x-x_0\right)\delta\left(y\right), \qquad (3)$$

where $\Delta_{x,y} = \partial^2/\partial x^2 + \partial^2/\partial y^2$, and the superscript 0 indicates the zero approximation to the potential. The solution to Eq. (3) with the zero potential boundary condition is

$$\varphi_e^{(0)} = \ln \frac{\sinh^2 \frac{\pi y}{4b(z)} + \cos^2 \frac{\pi(x+x_0)}{4b(z)}}{\sinh^2 \frac{\pi y}{4b(z)} + \sin^2 \frac{\pi(x-x_0)}{4b(z)}}. \qquad (4)$$

In the next approximation, the potential will be given by $\varphi_e = \varphi_e^{(0)} + \varphi_e^{(1)}$, where $\varphi_e^{(1)}$ is a first order correction that satisfies the following equation

$$\Delta_{x,y}\varphi_e^{(1)} = -\frac{\partial^2\varphi_e^{(0)}}{\partial^2 z^2}, \qquad (5)$$

with the boundary condition $\left.\varphi_e^{(1)}\right|_{x=\pm b(z)} = 0$. The solution to Eq. (5) can be found explicitly using $\varphi_e^{(0)}$ as a Green's function,

$$\varphi_e^{(1)}\left(x,y,x_0,z\right) = -\frac{1}{4\pi}\int\limits_{-b(z)}^{b(z)} dx' \int\limits_{-\infty}^{\infty} dy'$$

$$\times \frac{\partial^2\varphi_e^{(0)}}{\partial z^2}\left(x',y',x_0;b\left(z\right)\right)\varphi_e^{(0)}\left(x,y-y',x';b\left(z\right)\right). \qquad (6)$$

It is easy to show that the integral over y' in Eq. (6) converges on a scale of the order of $|y'| \sim b$ which, by assumption, is much smaller than the half width of the collimator h. This observation justifies the limit $h \to \infty$ assumed above.

2.2 *Magnetostatic Problem*

A specific feature of the magnetostatic problem is that even though $h \gg b\left(z\right)$, one cannot set $h \to \infty$ and consider an infinitely wide collimator. As we will see below, the contribution of the magnetic field into impedance has a term that is directly proportional to the width of the collimator h, and hence diverges in the limit $h \to \infty$. The physical mechanism of this divergence is related to the currents generated in the walls of the collimator due to the variation of the image charges. This current flows around the collimator cross section, as shown in Fig. 2, and generates the magnetic field H_y which turns out to be proportional to h.

It follows from the Maxwell equations that the y component of the magnetic field, H_y, of an infinitely thin current wire between perfectly conducting walls satisfies the following equation

$$\Delta H_y = 4\pi\delta'\left(x-x_0\right)\delta\left(y\right), \qquad (7)$$

with the Neumann boundary condition at the upper and lower walls, $\left.\partial H_y/\partial n\right|_{x=\pm b(z)} = 0$, and the Dirichlet boundary condition $H_y = 0$ at the lateral walls $y = \pm h$. It is convenient to formulate the magnetic problem in a way analogous to the electric one by introducing the magnetic "potential" φ_m such that $H_y = -\partial\varphi_m/\partial x_0$. Note that the derivative in this equation is taken with respect to the location of the driving particle x_0, rather than x. The potential φ_m satisfies the Poisson equation

$$\Delta\varphi_m = 4\pi\delta\left(x-x_0\right)\delta\left(y\right), \qquad (8)$$

with the boundary conditions

$$\left.\frac{\partial\varphi_m}{\partial n}\right|_{x=\pm b(z)} = 0, \quad \left.\varphi_m\right|_{y=\pm h} = 0. \qquad (9)$$

Again, invoking a perturbation theory, in the zero approximation which neglects the z-dependence in the potential φ_m, we have

$$\Delta_{x,y}\varphi_m^{(0)} = 4\pi\delta\left(x-x_0\right)\delta\left(y\right). \qquad (10)$$

The first of the boundary conditions (9) in this approximation takes the form

$$\left.\frac{\partial\varphi_m^{(0)}}{\partial x}\right|_{x=\pm b} = 0. \qquad (11)$$

The solution to this problem valid in the limit $h \gg b\left(z\right)$ is

$$\varphi_m^{(0)} = \ln\left[16\left(\sinh^2\frac{\pi y}{4b\left(z\right)} + \cos^2\frac{\pi\left(x+x_0\right)}{4b\left(z\right)}\right)\right.$$
$$\left.\times\left(\sinh^2\frac{\pi y}{4b\left(z\right)} + \sin^2\frac{\pi\left(x-x_0\right)}{4b\left(z\right)}\right)\right] - \frac{\pi h}{b\left(z\right)}. \qquad (12)$$

This function satisfies Eqs. (10) and (11), and is exponentially small at $y = \pm h$, $\left.\varphi_m\right|_{y=\pm h} \sim \exp\left(-\pi h/4b\right)$. For any practical purposes, the second of the boundary conditions (9) can be considered as satisfied as soon as $h > 4b$. Note that the presence of the second term on the right hand side of Eq. (12) does not allow us to consider the limit $h \to \infty$ in the magnetostatic problem.

In the next approximation, $\varphi_m = \varphi_m^{(0)} + \varphi_m^{(1)}$, where $\varphi_m^{(1)}$ satisfies

$$\Delta_{x,y}\varphi_m^{(1)} = -\frac{\partial^2\varphi_m^{(0)}}{\partial^2 z^2}. \qquad (13)$$

However, the boundary condition for the function $\varphi_m^{(1)}$ differs from the zero approximation (9), because the normal to the wall $\mathbf{n}$ in the first approximation is equal to $\mathbf{n} = \left(\pm 1, 0, -b'\right)$, yielding the boundary condition

$$\left.\frac{\partial\varphi_m^{(1)}}{\partial x}\right|_{x=\pm b} = \pm b'\left(z\right)\left.\frac{\partial\varphi_m^{(0)}}{\partial z}\right|_{x=\pm b}. \qquad (14)$$

The solution of the inhomogeneous equation (13) with the Neumann boundary condition (14) can be explicitly expressed in terms of the Green's function $\varphi_m^{(0)}$ [2] similar to Eq. (6). The result can be found in Ref. ([3]).

3 IMPEDANCE

The impedance is given by Eq. (1),

$$Z_x = -\frac{i}{c}\int_{-\infty}^{\infty}\left(\frac{\partial\varphi_e^{(0)}}{\partial x}+\frac{\partial\varphi_e^{(1)}}{\partial x}-\frac{\partial\varphi_m^{(0)}}{\partial x_0}-\frac{\partial\varphi_m^{(1)}}{\partial x_0}\right)dz. \tag{15}$$

Note that from Eqs. (4) and (12) it follows that

$$\partial\varphi_e^{(0)}\Big/\partial x = \partial\varphi_m^{(0)}\Big/\partial x_0, \tag{16}$$

which means that the zero order terms do not contribute to the impedance. This fact can be easily explained by that in the zero order theory the geometry of the collimator reduces to the rectangular pipe of a constant cross section, in which the wake of an ultrarelativistic beam is known to be zero. In the first approximation we have,

$$\int_{-\infty}^{\infty}(E_x - H_y)\,dz = -\int_{-\infty}^{\infty}\left(\frac{\partial\varphi_e^{(1)}}{\partial x}-\frac{\partial\varphi_m^{(1)}}{\partial x_0}\right)dz. \tag{17}$$

Near the axis when $x, x_0 \ll b$ one can expand the general expressions for the impedance and carry out the integration over y. Omitting lengthy analytical calculations that were performed with the use of computer program Mathematica [4] we present here the result. We limit our consideration to the case $y = 0$ only, i.e. the case when both the driving and test particles are in the same vertical plane. In this case,

$$Z_x = \frac{-iZ_0}{4\pi}\left(Ax + Bx_0\right), \tag{18}$$

where

$$A = 2I_1, \quad B = -2I_1 + 2\pi h I_3, \tag{19}$$

and

$$I_1 = \int dz\frac{(b')^2}{b^2}, \; I_2 = \int dz\frac{(b')^2}{b^3}. \tag{20}$$

For $x = x_0$, for a conventional definition of the transverse impedance Z_t, we find

$$Z_t \equiv \frac{1}{x_0}\left. Z_x\right|_{x=x_0} = \frac{-iZ_0}{2}hI_3. \tag{21}$$

Note that the integral I_1 also appears in Yokoya's theory of a smooth axisymmetric collimator [5], where $b\,(z)$ plays a role of the pipe radius. Comparison of our result with [5] shows that the impedance of a flat collimator is much larger (by factor of h/b) than the impedance of a cylindrical collimator of a radius $b\,(z)$.

4 NLC-TYPE COLLIMATOR

We apply the results obtained above to collimators considered in the design of the Next Linear Collider [6]. The geometry of a typical collimator is shown in Fig. 3 with the following parameters: $b = 0.5$ cm, $g = 0.1$ cm, $l = 40$ cm,

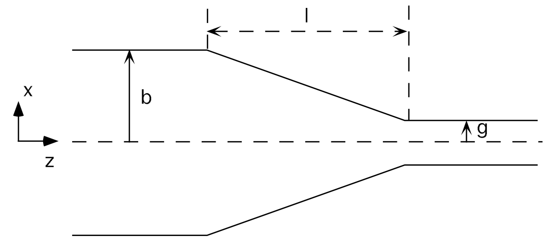

Figure 3: Schematic of the NLC collimator.

and the width of the collimator $h = 0.7$ cm. The applicability condition of the theory developed in the previous sections require the function $b\,(z)$ to be smooth together with its first two derivatives. Strictly speaking, this requirement does not hold for the profile shown in Fig. 3 where $b'\,(z)$ is not continuous at the entrance and the exit of the taper. In our calculations we assumed that in reality the angles of the collimator will be rounded in such a way that the smoothness condition is satisfied.

The integrals (20) for this collimator are equal: $I_1 = 0.08\ \mathrm{cm}^{-1}$ and $I_2 = 0.48\ \mathrm{cm}^{-2}$, which gives the transverse impedance near the axis,

$$-\mathrm{Im}Z_t = 6.3\times 10^3\ \mathrm{Ohm/m}. \tag{22}$$

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